Nonlinear and Adaptive Signal Estimation

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Abstract— **Kalman filter and its variations are commonly used to estimate signals observed under noise. The estimator is typically derived by minimizing the squared error between the observed and the estimated signals, and leads to a closed form optimal solution under Gaussian noise. However, such an approach leads to a solution where the Kalman gain is updated based on known noise statistics alone and not on the actual error. Hence the estimation could deviate from the actual if the noise statistics is different or time-varying, and could become unstable as well. In this paper we present a new Nonlinear and Adaptive signal estimator that introduces a closed loop between the error estimation and the gain adaptation and leads to an inherently robust estimator. Also, since nonlinear adaptive technique is utilized, complex error norms can as well be used. The design methodology basically involves making the combined estimator and the gain-update dynamics to resemble the dynamics of a nonlinear time varying (NLTV) electrical circuit having the required properties. This conceptually simple procedure leads to a new general class of complex nonlinear and adaptive estimation algorithms without the use of or the necessity for complex analytical tools, and hence can be understood and applied easily. We present below the basic methodology and simulation results to show the simplicity and the strength of the new approach**.

I. INTRODUCTION

In this paper, we consider the problem of signal estimation from a new and interesting perspective. The classical analytical approach to solve this problem is to define a suitable error function, mostly the squared error, take the derivative, set it to zero, and solve for the estimator parameters. Kalman filtering [1] is a good example of this approach. Here we look at systems for signal estimation, and in fact all NLTV or non-autonomous systems from the perspective of implementation as electrical circuits. The new paradigm is based on a bottom-up approach for system design, where we start with the elements (or in fact their I/O mappings) and proceed to form circuits and the general form of the dynamics in terms of the I/O mappings of the various elements. Such generic equations with the appropriate conditions are then used as templates for further design. As we will show in this paper, the new approach makes the design of complex NLTV systems and their applications to a number of problems relatively an easy job. In this paper, we apply it to signal estimation, and in another paper, we discuss the application to nonlinear control of helicopters.

Key concepts such as power, energy etc. that are unique to the analog world are used here to define the various NLTV electrical elements. Implementation of such devices and circuits in the analog domain is very complicated. Luckily, digital implementation has become the preferred mechanism for most applications, and that is what we propose here. We will derive the required dynamics in the analog domain, and transform it to the digital domain using proper A/D transformations. Also note that the basic properties such as loss or lossless can be preserved more easily in the digital domain even when finite precision arithmetic is used. Hence, we use the best of both analog and the digital world, in the solution proposed here.

The method proposed here is some what similar to solutions based on concepts such as passivity, positive real (PR) functions, and dissipative system models (such as Lagrangian and Hamiltonian Systems) that have long been used in areas such as I/O stability and controller design. However, the previous research uses those concepts from a macro or global perspective whereas we work at a micro or element level. Even though the distinction is very subtle, the new approach provides us the flexibility to tailor the circuit / dynamics to the application of interest and also eliminates the necessity to deal with analytical tasks such as finding the Lyapunov function etc. The solutions for different applications based on passivity concepts given in a recent book, "Nonlinear control for under-actuated mechanical systems," [2] illustrate the complexity that we face when we work at the macro level. The solution for estimation given here and the helicopter control should demonstrate the superiority of the new approach. Our method is similar to reverse engineering where we try to make a new system by studying the product from a competitor. We learn from electrical circuits (formed on a piece of paper) and use that knowledge to design electrical as well as non-electrical systems for various applications. At the same time, we try to maintain the internal architecture.

II. SIGNAL ESTIMATION USING KALMAN FILTERS

In this section, we briefly discuss the basis of signal estimation using Kalman filtering. The objective here is to provide a very brief introduction and to point out the missing link in the estimator algorithm and pave the way for the new approach. We start with the known model of the plant:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\,\mathbf{x}(t)+\mathbf{b}\,u(t)+\mathbf{G}\,\mathbf{v}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,x(t)+w(t) \end{aligned} \qquad (1\text{–}2)$$

Where **x**(t) is the unknown state vector and **y**(t) is the measured output vector. The plant dynamics and the output measurements are assumed to be corrupted by two white, zero-mean, mutually uncorrelated noise signals **v**(t) and **w**(t) with covariance matrices **V** and **W**. We also assume that we have an estimate of the initial state **x**(0) that is uncorrelated with the noises and with a known covariance matrix **P**(0). The task is to get an optimal estimate (denoted as $\hat{\mathbf{x}}$) of the state vector given the noisy output vector. The standard procedure is to define an estimate and a squared error:

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\,\hat{\mathbf{x}}(t)+\mathbf{b}\,u(t)+\mathbf{K}(t)\left[\mathbf{y}(t)-\mathbf{C}\,\hat{\mathbf{x}}(t)\right] \\ E\left[\mathbf{e}_x^t(t)\,\mathbf{e}_x(t)\right] &= tr\left\{E\left[\mathbf{e}_x(t)\,\mathbf{e}_x^t(t)\right]\right\} = tr[\mathbf{P}(t)] \end{aligned} \qquad (3\text{–}4)$$

where $\mathbf{y}(t)-\mathbf{C}\,\hat{\mathbf{x}}(t) = \mathbf{e}_y(t) = \mathbf{C}\,(\mathbf{x}(t)-\hat{\mathbf{x}}(t)) = \mathbf{C}\,\mathbf{e}_x(t)$ and $\mathbf{e}_x$ is the error in the state estimate. The Kalman gain matrix **K**(t) in the above estimate is found by minimizing the squared error for all time t leading to the solution:

$$\begin{aligned} \dot{\mathbf{P}}(t) &= \mathbf{A}\,\mathbf{P}(t)+\mathbf{P}(t)\mathbf{A}^t - \mathbf{P}(t).\mathbf{C}^t\,\mathbf{W}^{-1}\mathbf{C}.\mathbf{P}(t) \\ \mathbf{K}(t) &= \mathbf{P}(t)\mathbf{C}^t\,\mathbf{W}^{-1} \end{aligned} \qquad (5\text{–}6)$$

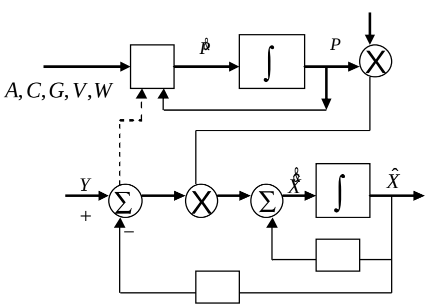

Fig 1 : Block Diagram of a Kalman Estimator.

A block diagram of the Kalman estimator is shown in Fig. 1. Note that the Kalman gain is updated using the system parameters and the noise statistics which in turn is used in the signal estimation. That is, the process involves a feedforward architecture and the error in the output estimate which is readily available is not used in Kalman gain update. This is a consequence of the analytical approach. By providing a feedback from the error to the gain update as shown in the same figure by a dotted line we can have a fully coupled robust estimator. Unfortunately, designing such an estimator using analytical techniques is not an easy task even for LTI systems. In section IV, we discuss the design of such an estimator regardless of whether the plant is LTI or NLTV. First let us provide a brief introduction to the building block approach that is the principle behind our design technique.

III. NLTV CIRCUIT ELEMENTS, CIRCUITS, DYNAMICS, AND RESULTING RESPONSES

In this section, we will briefly discuss the physical models of some NLTV circuit elements and the dynamics / responses from circuits made from such elements. Details can be found elsewhere [3]. Let us first discuss the family of *resistors* using a two-port resistor. Letting $\mathbf{v} = [v_1 \quad v_2]^t$ and $\mathbf{i} = [i_1 \quad i_2]^t$ as the voltage and current vectors, we can write two models, *voltage controlled* and *current controlled* as:

$$\begin{aligned} &v_1(\mathbf{i},\alpha) = v_1[\mathbf{i}(t),\ \alpha] \ \&\ v_2(\mathbf{i},\alpha) = v_2[\mathbf{i}(t),\ \alpha] \\ &i_1(\mathbf{v},\alpha) = i_1[\mathbf{v}(t),\ \beta] \ \&\ i_2(\mathbf{v},\alpha) = i_2[\mathbf{v}(t),\ \beta] \end{aligned} \qquad (7)$$

The parameters α and β are used to lump the effects of the various phenomena that change as a function of time. The power p(t) going into the multi-port resistor is given by $\mathbf{v}^t\,\mathbf{i}$. The resistor is a *static* device and by constraining the equations, we can arrive at various categories. The element is said to be *fully passive* if p(t) ≥ 0 for all time t, active if p(t) ≤ 0 for all t, and non-passive otherwise. If the **v-i** relationship in (7) can be written in the form $\mathbf{v} = \mathbf{R}[\alpha]\mathbf{i}$, we have a *separable LTV model*; $\mathbf{R}[\alpha]$, the time-varying resistive matrix is time-varying positive definite (or semi) for passive resistors; if $\mathbf{v} = \mathbf{R}[\alpha]\mathbf{f}[\mathbf{i}]$, where $\mathbf{f}[\mathbf{i}]$ is a nonlinear mapping of the input vector, we have a *separable NLTV model*; if the effect of time and the inputs cannot be separated, we have *non-separable models*; if the mapping is a many to one, we have *non-invertible mapping* implying that the input and output designations need to be preserved. Some specific examples are:

$$\begin{aligned} &v(i,\alpha) = R[\alpha(t)]i(t)\ \textit{with}\ 0<R_{\min}<R[\alpha]<R_{\max} \\ &v(i,\alpha) = R[\alpha(t)]\ i^3(t) \\ &v(i,\alpha) = i(t)\left[i^2+2\,i\sin(t)+1\right] \end{aligned} \qquad (8)$$

NLTV Inductors: We can visualize a NLTV inductor by considering its implementation and by letting some of the physical quantities vary within the allowed range and derive the properties from physical considerations. We can have a current controlled inductor $\phi = \phi_L[i_L,\alpha]$, or a flux controlled inductor, $i_L = i_L[\phi,\alpha]$. At any given time, the NLTV inductor represents a NLTI inductor and hence has to have all the properties of an inductor such as no loss, energy storage and return (memory property). Hence considering a flux controlled NLTV inductor, and assuming $\phi_{rel} = 0$ as the relaxation point or the value at which there is no energy left in the inductor, we get:

$$E_L[\phi = 0, \alpha] = 0 \;\; and \;\; E_L[\phi \neq 0, \alpha] > 0 \quad for\ all\ \alpha \qquad (9\text{–}10)$$
$$E_{low}[\phi] \leq E_L[\phi, \alpha] \leq E_u[\phi]$$

where, $E_{low}[\phi]$ and $E_u[\phi]$ are two positive definite functions. We arrive at the first constraint because no flux implies no stored energy regardless of the value of α. The second constraint follows from the fact that we have an energy storage element. The third condition says the energy function of a NLTV inductor must be a locally positive definite and a decrescent function. That is, the stored energy has to be bounded by some PD functions of the flux alone. It follows from the fact that the values of the physical parameters are bounded, and for any fixed value of α or time t, the device represents an inductor. We can also let $E_L[\phi, \alpha] \to \infty$ as $|\phi| \to \infty$ as is the case in LTI inductors so that we can talk of the properties in a global sense. From the energy expression, we can obtain an expression for the voltage as $i_L(t) = \mathrm{d}E_L[\phi, \alpha]/\mathrm{d}\phi$. Similar to the case of resistors, we can obtain LTV, separable NLTV and non-separable NLTV models as shown in the examples below:

$$E_L[\phi, \alpha] = \{1 + \sin^2(t)\}\phi^2(t); \;\; i_L(t) = 2\{1 + \sin^2(t)\}\phi(t)$$
$$E_L[\phi, \alpha] = \{1 + \sin^2(t)\}\phi^4(t); \;\; i_L(t) = 4\{1 + \sin^2(t)\}\phi^3(t)$$
$$E_L[\phi, \alpha] = \left(\phi^2 - 1\right)^2 \left\{\phi^2 - 2\phi \sin(t) + 1\right\}$$
$$i_L(t) = 4\phi\left(\phi^2 - 1\right)\left[\{\phi - \sin(t)\}^2 + \cos^2(t)\right] + 2\left(\phi^2 - 1\right)^2\{\phi - \sin(t)\}$$

In the last example, the inductor current becomes zero for three values of ϕ (± 1 and 0). Of these three values, the two values ± 1 are the relaxation points (zero stored energy) and $\phi = 0$ corresponds to a point of local maxima for the stored energy. Finally, it should be pointed out that the elements property $v_L = \dot{\phi}$, circuit interconnection laws (basically KCL and KVL), and circuit solvability considerations dictate that the ϕ *Vs* i_L characteristics be monotonic when i_L is the independent variable where as the i_L *Vs* ϕ characteristic can be more general when ϕ is the independent variable. Therefore, we will assume that NLTV inductors are always flux-controlled. In such a case, ϕ is the independent or the state variable, $i_L = i_L[\phi, \alpha]$ is a NLTV mapping of the state variable, and $v_L = \dot{\phi}$ is the voltage. Therefore, when a NLTV resistor is connected in series with an inductor, we need to make sure that the resistor is current controlled to ensure the solvability of the resulting dynamics.

We can form various circuits from such elements and obtain the general form of NLTV dynamics from the circuits. It should be clear that the dynamics will have NLTV terms that represent the mappings of valid electrical elements. By varying the mappings within the allowed domain for each of the elements, we can obtain different NLTV dynamics belonging to a particular family. Note that the reactive elements still play the role of energy storage and return (lossless, memory devices), but the energy envelope will be more complex than just the bow shaped energy curve of LTI memory elements. Further, they can have multiple relaxation or zero energy points and local minima and maxima points. The relaxation points and the points of local minima of the energy become the equilibrium points of the dynamics. The sum of the energy in all the reactive elements in the circuit is one good Lyapunov function. This total energy function resulting from the use of only inductors and or capacitors will be a separable function of the state variables. The use of mutual inductors will lead to non-separable energy functions.

The lossless, static multi-port Elements such as gyrators and transformers play the role of transferring power from some ports to the rest of the ports, and hence help to shape the required response. The NLTV resistors provide the necessary mechanism to consume power (if that is what is needed as in a stable dynamics) or could be used to consume and or deliver power depending on the value of the independent variable. We will discuss a specific family of circuits corresponding to signal estimation in the next section.

IV. APPLICATION TO SIGNAL ESTIMATION

Considering a NLTV plant with **x**, **y**, u respectively as the state, output, and input vectors of appropriate dimensions, $\hat{\mathbf{x}}$ as the estimated state vector, and **k** as the gain used in the estimation and represented as a column vector, we can write the estimation and gain update dynamics as:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}_{est}[\hat{\mathbf{x}}, \mathbf{y}, u, \mathbf{k}, t] \;\; \& \;\; \dot{\mathbf{k}}(t) = \mathbf{f}_{gain}[\hat{\mathbf{x}}, \mathbf{y}, u, \mathbf{k}, t] \qquad (11)$$

Letting $\mathbf{k}_\infty$ be the given value of **k** as $t \to \infty$, the various error vectors as $\mathbf{e}_x = \mathbf{x} - \hat{\mathbf{x}}$, $\mathbf{e}_y = \mathbf{y} - \hat{\mathbf{y}}$, $\mathbf{e}_k = \mathbf{k} - \mathbf{k}_\infty$, the dynamics can be written in terms of the error vectors as:

$$\dot{\mathbf{e}}_x(t) = \mathbf{f}_{\mathbf{e}\_x}\left[\mathbf{e}_x, \mathbf{e}_k, \mathbf{e}_y, t\right] \;\; \& \;\; \dot{\mathbf{e}}_k(t) = \mathbf{f}_{\mathbf{e}\_k}\left[\mathbf{e}_x, \mathbf{e}_k, \mathbf{e}_y, t\right] \; (12)$$

The combined error dynamics (in the absence of any noise in the plant) should have the origin as the globally uniformly asymptotically stable (UAS). From the circuit perspective, the dynamics should point to a NLTV circuit made of NLTV reactive elements, static multi-port lossless elements, and globally passive resistors (that is, the v-i characteristics should be confined to the first and the third quadrants). Further, all the reactive elements must have their relaxation points at and only at the origin, and have monotonically increasing energy storage functions with the energy becoming unbounded as the independent variables become unbounded. Hence we can put together such a circuit on a piece of paper, write down the corresponding dynamics in terms of the I/O mappings of the various elements, reconcile it with the plant dynamics and the chosen error function to arrive at the estimation and gain update dynamics. Thus we can obtain a family of such dynamics where we can have complex error measures and obtain superior estimation. (In our earlier work [4], we tried to mimic the Kalman approach where we first update the noise statistics **P** using a dynamic equation and find the gain matrix from it making the implementation complex. In this paper, we write the dynamics in terms of the gain itself which makes it possible to design the estimator rather easily regardless of whether the plant is LTI or NLTV)

The method discussed above would require knowledge of the noisy state vector to update the gain vector. If only the output vector is available, we need to use a circuit architecture like the one discussed above driven by constant valued sources and write the dynamics directly in terms of the estimated and gain vectors and not the error vectors. We will explain this through the estimator design for a specific 2nd order LTI plant [5] given by:

$$\mathbf{x} = \begin{bmatrix} -4 & 2 \\ -2 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 1 \\ -1 \end{bmatrix}v(t) \qquad (13\text{–}14)$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} + w$$

A family of estimator and gain dynamics for this plant can be written from circuit considerations as:

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\mathbf{k}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} = -\mathbf{L}_1^{-1}\mathbf{Z}_{p11} & \mathbf{A}_{12}[\mathbf{e}_y] = -\mathbf{L}_1^{-1}\mathbf{Z}_{g12} \\ \mathbf{A}_{21}[\mathbf{e}_y] = \mathbf{L}_2^{-1}\mathbf{Z}_{g12}^t & -\mathbf{L}_2^{-1}\mathbf{Z}_{p22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \mathbf{k} \end{bmatrix} + \begin{bmatrix} \mathbf{b}u \\ \mathbf{k}_{ss} \end{bmatrix} \qquad (15)$$

from a passive circuit having the required properties as shown in Fig. 2 (without the sources). To obtain a particular solution, we will fix $\mathbf{z}_{p11}$ as non-symmetric PD and solve for $\mathbf{L}_1$ from $\mathbf{L}_1 = \mathbf{Z}_{p11}\mathbf{A}^{-1}$. $\mathbf{A}_{12}[\mathbf{e}_y]$ comes from the weighted error function used in the estimation dynamics. We can use any form of error such as integration over a specific period weighted by the gain parameters. From $\mathbf{L}_1$, $\mathbf{A}_{12}[\mathbf{e}_y]$, assumed value for $\mathbf{L}_2$ (must be PD) and $\mathbf{Z}_{g12}$ (which could be any arbitrary matrix), we will obtain $\mathbf{A}_{21}[\mathbf{e}_y]$. $\mathbf{Z}_{p22}$ can be chosen arbitrarily but PD and $\mathbf{k}_{ss}$ is obtained by solving $\mathbf{L}_2^{-1}\mathbf{Z}_{p22}\mathbf{k}_\infty = \mathbf{k}_{ss}$. Note that while comparing with the circuit dynamics and solving for the various quantities, we fix the impedance matrix of the static passive network and solve for the inductance matrix, which should turn out to be PD. This is very similar to testing for stability of LTI systems using Kalman-Yakabovich lemma where for a given system matrix **A**, we assume arbitrary PD matrix **Q** and solve for the matrix **P** from $\mathbf{PA} + \mathbf{A}^t\mathbf{P} = -\mathbf{Q}$ and not the other way. In fact, if we let $\mathbf{x} = \mathbf{i}_L$, $\mathbf{P} = \mathbf{L}$, and $\mathbf{Q} = \mathbf{Z}_r$, we will find that $\mathbf{x}^t\mathbf{P}\mathbf{x} = E_L$, the energy in left in the mutual inductor, and $\mathbf{x}^t\mathbf{Q}\mathbf{x} = -0.5\mathbf{x}^t\left(\mathbf{PA} + \mathbf{A}^t\mathbf{P}\right)\mathbf{x}$ is the power leaving the mutual inductor and has to be positive since the rest of the circuit is passive. Anyway, returning to the signal estimation problem, we picked various quantities in the dynamics almost randomly (subject to the given constraints), used a 16 point average of $\mathbf{e}_y$, and simulated both the Kalman and the new estimator. The error becomes almost zero for the new estimation algorithm. Simulation results will be shown at the conference. The new estimator is also robust to variations in the signal statistics and the chosen estimator parameters. The Kalman gain reaches the steady state value very quickly. The new gain update uses a highly coupled NLTV dynamics that responds in a proper way to eliminate the error in the estimation.

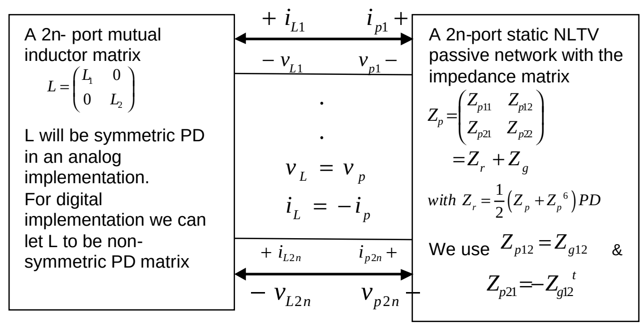

Fig 2 : A Passive Circuit leading to a family of NLTV Signal Estimators.

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