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CHAOTIC DYNAMICS, CHAOS SYNCHRONIZATION, STATE ESTIMATION, AND INTERPRETATION USING ELECTRICAL ELEMENTS AND CIRCUITS

P. A. Ramamoorthy

[1]Department of Electrical & Computer Engineering and Computer Science
University of Cincinnati, Cincinnati, OH 45221

pramamoo@ececs.uc.edu ; paramamoorthy@fuse.net

Abstract. Chaotic systems and chaotic signal synchronization have attracted a lot of attention lately. Most work on chaotic systems is based on few well-known 3^{rd}-order models derived using analytical techniques. In addition, chaotic signal synchronization or the inability to achieve the same is proven using analytical concepts such as Lyapunov constants etc. that could be understood only by those mathematically gifted. Such methods also have limitations such as the initial state of the coupled chaotic systems cannot be too far away etc. Most of these problems are associated with the application of complex analytical techniques to an area which itself is too complex, forcing us to settle for special models, assumptions, etc. In this paper, we consider two major problems, 1) Design of Chaotic Systems, and 2) Chaos synchronization, from an electrical circuits' perspective, and show how an elegant solution to these two problems can be achieved from a new approach that we call as the building block approach. This approach minimizes the need for complex analytical techniques (or moves the complexity away from a macro or global level to a micro or local level) in analyzing and or designing such systems, and more importantly, gives us the ability to concentrate more on what we can achieve from such complex systems. We will also cast the problem of chaotic signal synchronization in the form of state estimation given partial state observation (except that here we are dealing with a transmitter that is a complex quasi-stable system) and hence the solution that we propose here applies to the general problem of state estimation as well.

Keywords. NLTV circuits and dynamics, chaos, chaos synchronization, state estimation

1 Introduction

In this paper, we discuss chaotic systems and chaotic signal synchronization, areas that have attracted a lot of attention lately [1], from a new and interesting perspective. Instead of looking at chaotic systems and synchronization from the classical analytical approach (dynamics to mathematical tools for characterization / stability testing to simulation and implementation), we look at such systems (and in fact, all nonlinear time-varying, NLTV, or non-autonomous systems) from an implementation (as electrical circuits) perspective. Thus, the thrust of our work is based on answers to a number fundamental questions: 1) To build complex NLTV circuits, what kind of electrical elements or building blocks we need and what are their properties? 2) What kind of elements we need to build chaotic circuits in particular? 3) If the process of chaotic signal synchronization is viewed from a circuits' perspective, what is the circuit realization of this problem? 4) Can the process of state estimation (from a partial state observation) be explained from a circuits perspective? 5) More importantly, can we use such elements, circuits formed (on a piece of paper) from such elements, and the dynamics resulting from such circuits as templates for further design, something similar to reverse engineering? The answers to these questions provide an elegant framework to handle the design of many systems in the nonlinear and time-varying domain. In this paper, we apply it to chaos system design, chaos synchronization, and the more general problem of state estimation. The new paradigm is shown in Fig. 1. As can be seen from the figure, it is a bottom-up approach for system design, where we start with the elements (or their definitions) and proceed to form circuits and the dynamics in terms of the I/O mappings of the elements. Such generic equations with the appropriate conditions are then used in real-world applications. Note that earlier approaches for chaotic system design or synchronization are based on couple of well known 3^{rd} order models and rely on complex analytical tools to show their behavior or to show that local or global synchronization can be achieved [2-4]. This new approach makes the understanding of such complex phenomena and the design of such complex systems much easier, and using this simple approach we can teach even undergraduate students the complex area of NL and TV systems, as this author has been doing for a number of years. We recognize that the concept of passivity has long been used in areas such as I/O stability, positive real (PR) functions, and dissipative systems using Lagrangian and Hamiltonian Systems. However, it is used from a macro perspective where as here we work at a micro or element level that provides us the flexibility to tailor the circuit / the dynamics to the application of interest. Further, our method is similar to reverse engineering where we try to make a new system by studying the product from a competitor, except that we are not breaking any laws here. We learn from electrical circuits (formed on a piece of paper) and use that knowledge to design electrical as well as non-electrical systems for various applications.

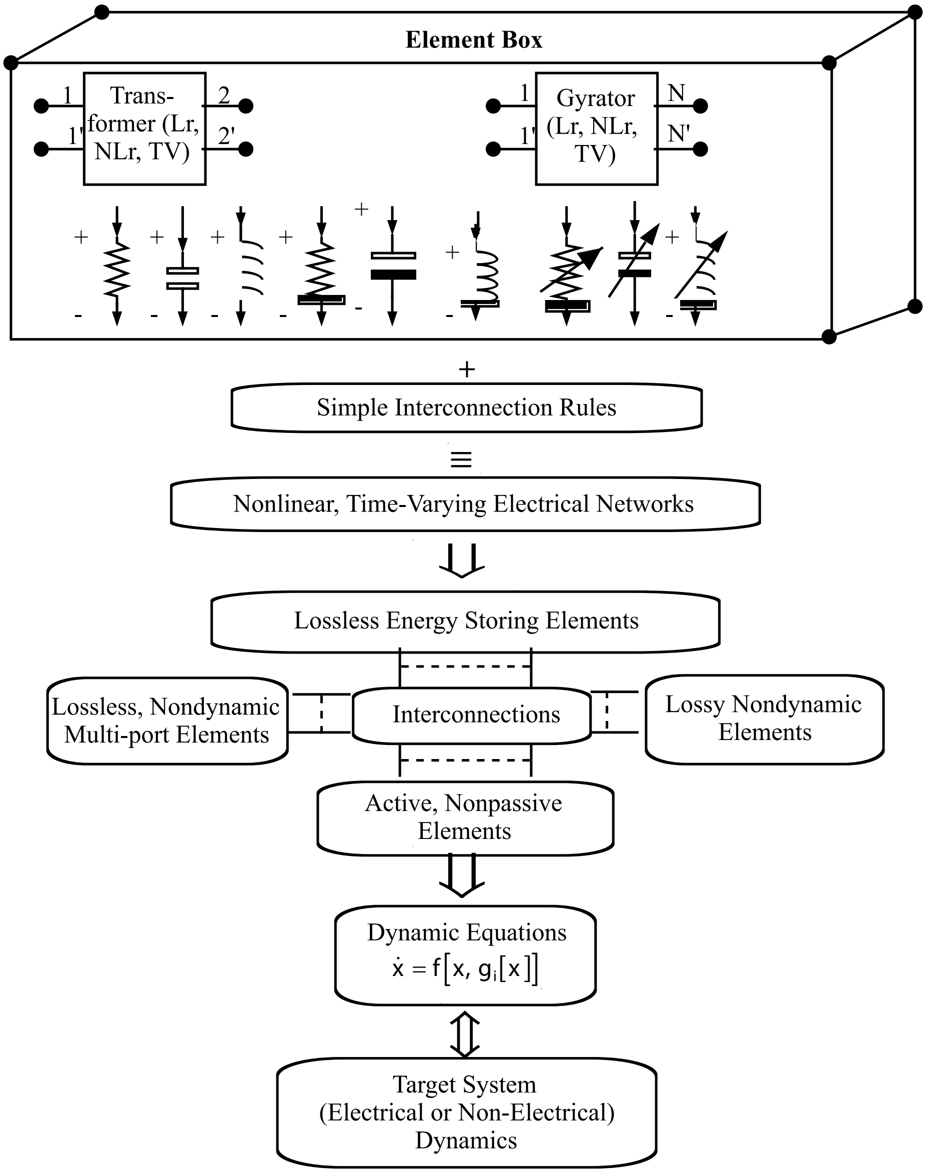

Figure 1: A new paradigm for nonlinear (autonomous & non-autonomous) dynamic systems design

2 Nonlinear & Time-varying Circuit Elements

In this section, we will briefly discuss the models for various NLTV electrical elements for building complex NLTV circuits[1]. The models will be arrived looking at physical realizability. Let us first consider the case of a *NLTV resistor*, a simple *static* device. We can write two generic models, one as *current controlled*, and another as *voltage controlled*:

$$v[t,\alpha] = v[i[t],\alpha] \quad or \quad i[t,\beta] = i[v[t],\beta] \tag{1}$$

The parameters α and β are used to lump the effects of the various physical phenomena that are assumed to change as a function of time. We can obtain different models by looking at possible realizations. For example, a *separable passive LTV model* can be obtained from a rheostat as:

$$v[t,\alpha] = R[\alpha(t)]\, i(t) \quad with \quad 0 < R_{min} \le R(t) \le R_{max} \tag{2}$$

where $R[\alpha]$ represents the time varying resistance and points to a resistor that is passive and consumes minimum amount of power at any given time. We can make it nonlinear by letting the current to voltage relationship nonlinear. For example, we can have *separable and non-separable passive NLTV resistor* models:

$$v[i(t),\alpha] = R[\alpha]\, i^3(t) \tag{3}$$

$$v[i(t),\alpha] = i(t)\{i^2(t) + 2i(t)\sin(t) + 1\} \tag{4}$$

From the examples, we can note that $v[i(t)=0,\alpha] = 0$ for all values of α and $v[i(t),\alpha] > 0$ for $i(t) > 0$ and $v[i(t),\alpha] < 0$ for $i(t) < 0$ for *passive resistors*. The characteristics will be in the second and fourth quadrants of the $v(t)$ Vs $i(t)$ plane for *negative resistors* and can be in any quadrant for *non-passive resistors*. Also, note that the mappings can be many to one and hence we need to maintain the input and output designations. The type of resistors, current-controlled or voltage-controlled, that we use in a circuit will depend on the reactive elements and the type of connection. We will discuss this issue when we deal with reactive elements.

A NLTV gyrator is a static M-port (M $\geq$ 2) element with the I/O relationship (with $M = 2$):

$$\begin{bmatrix} v_1[t,\beta] \\ v_2[t,\beta] \end{bmatrix} = \begin{bmatrix} 0 & R_{12}[\mathbf{x},\beta] \\ -R_{12}[\mathbf{x},\beta] & 0 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \mathbf{R}[\mathbf{x},\beta] \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} \tag{5}$$

[1]Concepts such as power and energy are unique to the analog world. Hence, the analog or continuous-domain modelling is used here basically to arrive at the proper I/O mappings of various devices. Implementation of such devices and circuits is not an easy task as can be seen from the experience of researchers who worked on LTI gyrators and analog computers. Luckily, digital implementation has become the preferred mechanism for most applications, and that is what we propose here. We will derive the required dynamics in the analog domain, and transform it to the digital domain using proper A/D transformations. We can also preserve the basic properties such as lossless more easily in the digital domain even when finite precision arithmetic is used.

where $\mathbf{R}[\mathbf{x},\beta]$ is the resistive matrix of the gyrator and is a function of time as well as the state vector $\mathbf{x}$. Note that $\mathbf{R}[\mathbf{x},\beta] + \mathbf{R}^t[\mathbf{x},\beta] = \mathbf{0}$, which makes the power consumed by the gyrator zero all the time. That is, a NLTV gyrator is a *lossless static multi-port device* with a skew-symmetric resistor matrix $\mathbf{R}[\mathbf{x},\beta]$. Once again by proper choice of the matrix elements, we can arrive at specific classes of LTV, NLTI, separable, and non-separable models. The elements of the resistor matrix could be selected based on the application and won't allowed to become unbounded or indeterminant for finite values of the state.

NLTV Capacitors: We can visualize a NLTV capacitor by considering the realization of a capacitor and letting some of the parameters such as the dielectric constant ε and or the distance between the plates d be a nonlinear function of the voltage and or vary as a function of time subject to some physical constraints, $0 < \varepsilon_{\min} \le \varepsilon(t) \le \varepsilon_{\max}$ and $0 < d_{\min} \le d(t) \le d_{\max}$. For modelling purposes, we can lump the effects of the change in these two parameters into a single parameter $\alpha(t)$. We can have a *voltage-controlled capacitor*, $q_c(t) = q_c[v_c(t),\alpha(t)]$, or a *charge-controlled capacitor*, $v_c(t) = v_c[q_c(t),\alpha(t)]$. At any given time instance, the NLTV capacitor represents a NLTI capacitor and hence has to have all the properties of a capacitor such as losslessness, energy storage and return (memory property). Hence considering a charge controlled NLTV capacitor, and assuming $q_{cr} = 0$ as the relaxation point or the value at which there is no energy left in the capacitor, we get:

$$E_c[q_c = 0,\alpha] = 0 \quad and \quad E_c[q_c \neq 0,\alpha] > 0 \quad for\ all\,\alpha$$

$$E_L[q_c] \;\le\; E_c[q_c,\alpha] \;\le\; E_u[q_c] \tag{6}$$

where $E_L[q_c]$ *and* $E_u[q_c]$ are two positive definite functions. We arrive at the first constraint because no charge implies no stored energy regardless of what values $\varepsilon(t)$ and d(t) take. The second constraint follows from the definition of stored energy. The third condition means the energy function of a NLTV capacitor must be a locally positive definite and a decrescent function[2]. That is, the stored energy has to be bounded by some positive definite functions of the charge alone. It follows from the fact that the values of the parameters are bounded, and for any fixed value of α or time t, the device represents a capacitor. From the energy expression, we can obtain an expression for the voltage as $v_c(t) = \delta E_c[q,\alpha]/\delta q$. Similar to the case of resistors, we can obtain LTV, NLTI, separable, and non-separable models. Some examples are:

Linear time-varying capacitor:

$$E_c[q(t),\alpha(t)] = \{1+\sin^2(t)\}\,q^2(t); \quad v_c(t) = 2\{1+\sin^2(t)\}\,q(t) \tag{7}$$

Separable Nonlinear TV capacitor:

$$E_c[q(t),\alpha(t)] = \{1+\sin^2(t)\}\,q^4(t); \quad v_c(t) = 4\{1+\sin^2(t)\}\,q^3(t) \tag{8}$$

[2] We can also let $E_c[q_c,\alpha(t)] \to \infty$ *as* $|q| \to \infty$ as is the case in LTI capacitors so that we can talk of the properties in a global sense.

Non-separable NLTV capacitor with equilibrium points at $q_{cr} = \pm 1$:

$$\begin{aligned} E_c\left[q, \alpha\right] &= \left(q^2 - 1\right)^2 \left\{q^2 - 2q\sin(t) + 1\right\} \\ &= \left(q^2 - 1\right)^2 \left[\left\{q - \sin(t)\right\}^2 + \cos^2(t)\right] \end{aligned} \tag{9}$$

$$v_c = 4q\left(q^2 - 1\right) \left[\left\{q - \sin(t)\right\}^2 + \cos^2\right] + 2\left(q^2 - 1\right)^2 \left\{q - \sin(t)\right\} \tag{10}$$

In the last example, the capacitor voltage becomes zero for three values of q (± 1 and zero). Of these three values, the two values ± 1 are the relaxation points (zero stored energy) and q = 0 corresponds to a point of local maxima for the stored energy. Finally, it should be pointed out that the element property $i_c(t) = \dot{q}(t)$, circuit interconnection laws (basically KCL and KVL), and circuit solvability considerations dictate that the $q(t)\ Vs\ v_c(t)$ characteristics be monotonic when $v_c(t)$ is the independent variable where as the $v_c(t)\ Vs\ q(t)$ characteristic can be more general when q(t) is the independent variable. Therefore, we will assume that the NLTV capacitors are always charge-controlled. In such a case, q(t) is the independent or the state variable, $v_c\left[q(t), t\right]$ is a NLTV mapping of the state variable, and $i_c(t) = \dot{q}(t)$ is the current. Therefore, when a NLTV resistor is connected in parallel with a capacitor, we need to make sure that the resistor is voltage controlled to ensure the solvability of the resulting dynamics.

A time-varying inductor, being the dual of a capacitor, can be defined in a similar manner. We can also define ideal NLTV transformers similar to ideal LTI transformers and NLTV mutual inductors. Hence, we will omit the details. See ref. [5] for details.

3 NLTV Electrical Circuits, Dynamics, and Resulting Responses

We can form various circuits from the building blocks defined in the previous section and obtain the general form of NLTV dynamics from the circuits. It should be clear that the dynamics will have nonlinear and time-varying terms that represent the mappings of valid electrical elements. By varying the mappings within the allowed domain for each of the elements, we can obtain all possible NLTV dynamics belonging to a particular family. Note that the reactive elements still play the role of energy storage and return (lossless, memory devices), but the energy envelope will be more complex than just the bowel shaped energy curve of LTI memory elements. Further, they can have multiple relaxation or zero energy points and local minima and maxima points. The relaxation points and the points of local minima of the energy become the equilibrium points of the dynamics. The sum of the energy in all the reactive elements in the circuit is one good Lyapunov function. This total energy function resulting from the use of only inductors and or capacitors will be a separable function of the state variables. The use of mutual inductors will lead to non-separable energy functions.

The lossless, static multi-port Elements such as gyrators and transformers play the role of transferring power from some ports to the rest of the ports, and hence help to shape the required response. The NLTV resistors provide the necessary mechanism to consume power (if that is what is needed as in a stable dynamics) or could be used to consume and or deliver power depending on the value of the independent variable. We will now look at some general circuits made of specific category of elements and state the properties of resulting dynamics / their equilibrium points[3].

Type 1 Circuits: NLTV circuits made of NLTV reactive elements, static multi-port lossless elements, and resistors that are globally passive (that is, the voltage to current characteristic is confined to the first and the third quadrant). Further, let the reactive elements have their relaxation points at the origin only and have monotonically increasing energy storage functions with the energy becoming unbounded as the independent variables become unbounded: $\Rightarrow$ The resulting dynamics will have only one equilibrium point at the origin which will be *globally uniformly asymptotically stable* (UAS). A Lyapunov function for the dynamics will be the sum of energy stored in the various reactive elements. The derivative of the LF along the system trajectory will be the negative of the sum of power consumed by the resistors, and will always be negative for globally passive resistors. The shape of the resistor characteristics determines whether the equilibrium point is *exponentially* stable or not, and if the dynamics is *totally stable* (stability under persistent disturbances). Couple of simple examples will illustrate these points.

Consider a circuit with a single LTI capacitor of value 1 F, a NLTV voltage controlled resistor given by $i_R\left[v_R, t\right] = v_R^3(t)\left(1+\sin^2(t)\right)$ and a current source $i_s(t)$ all connected in parallel. Using KCL & KVL, and the elements' equations, we get circuit dynamics as: $\dot{v}_c(t) = -v_R^3(t)\left(1+\sin^2(t)\right) + i_s(t)$. We can note that this dynamics (with $i_s(t) = 0$) has only one equilibrium point at the origin that is globally UAS. However, the equilibrium point is not exponentially stable as the power consumption capacity of the resistor becomes negligible $\left(p_R(t) = v_R^4(t)\left(1+\sin^2(t)\right) = v_c^4(t)\left(1+\sin^2(t)\right)\right)$ when $|v_c(t)| < 1$. Similarly, if we replace the resistor by one with a characteristic $i_R\left[v_R, t\right] = \tanh\left[v_R(t)\right]$, the resulting dynamics will have the origin as the only exponentially stable equilibrium point. However, we can note that the dynamics will not be totally stable since the power absorption capacity is

[3] Given the complexity of NLTV elements and the circuits that can be made, it should be noted that the properties stated here are valid from a practical or conservative design perspective. However, we can also arrive at dynamics which have the same properties, but pointing to a different category of circuits. As an example, we can combine a number of non-passive resistors to come up with the characteristics of a globally passive resistor circuit.

reduced to $p_R(t) = |v_R(t)| = |v_c(t)|$ as $|v_c(t)| >> 1$. These two simple examples demonstrate the power of looking at NLTV dynamics not just from a mathematical perspective but as one from a circuit made of real elements.

Type 2 Circuits: NLTV circuits made of NLTV reactive elements with *multiple relaxation points and or multiple local minimal energy points*, static multi-port lossless elements, and resistors that are globally passive: $\Rightarrow$ The resulting dynamics will have multiple equilibrium points, some of which will be locally UAS, and some will be unstable. Whether the stable equilibrium points are exponentially and or totally stable will depend on the resistor characteristics.

Type 3 Circuits: NLTV circuits made of NLTV reactive elements with *multiple relaxation points and or multiple local minimal energy points*, static multi-port lossless elements, and resistors whose properties change from passive to non-passive or vice versa based on the input values $\Rightarrow$ The properties of the resulting dynamics can be all over the map depending on when the resistors become passive and non-passive. These types of circuits can be configured properly to arrive at dynamics that exhibit limit cycle and chaotic behavior, as we will see in the next section.

4 Chaotic Dynamics, Chaos Synchronization, and State Estimation

In this section, we will discuss how the circuit approach can be applied in a simple manner to arrive at various chaotic dynamics, chaos synchronization, and the related problem of state estimation. Before we do that, we will first discuss the circuit equivalence of the *Kalman-Yakubovic (K-Y) lemma* used frequently to prove the stability or the lack of it of a LTI system. K-Y lemma states that given any positive definite matrix $\mathbf{Q}$ and a LTI system represented as $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$, the solution for $\mathbf{PA} + \mathbf{A}^t\,\mathbf{P} = -\mathbf{Q}$ should lead to a matrix $\mathbf{P}$ that is positive definite for the system to be stable. Consider the general architecture for NLTV circuits as shown in Fig.2 consisting of a n-port NLTV coupled inductor [4] and a n-port static passive network. For K-Y lemma, they will be reduced to LTI elements. Denote by ϕ, $\mathbf{i}_L$, $\mathbf{v}_L$ respectively the flux, current and voltage vectors, and $\mathbf{L}$ the mutual inductance matrix, a positive definite one, of the coupled inductor. Similarly, let $\mathbf{i}_p$, $\mathbf{v}_p$ be the current and the voltage vectors and $\mathbf{Z}_p$ the impedance matrix of the static, passive network, where $\mathbf{Z}_R = \mathbf{Z}_p + \mathbf{Z}_p^t$ has to be positive definite. Then, from the elements and interconnections, we can write, $\dot{\phi} = \mathbf{v}_L = \mathbf{L}\,\dot{\mathbf{i}}_L = \mathbf{v}_p = \mathbf{Z}_p\,\mathbf{i}_p = -\mathbf{Z}_p\,\mathbf{i}_L$ leading to $\dot{\mathbf{i}}_L = -\mathbf{L}^{-1}\,\mathbf{Z}_p\,\mathbf{i}_L$. Equating the state vector $\mathbf{x}$ of the LTI system to the coupled inductor current vector, we get $\mathbf{A} = -\mathbf{L}^{-1}\,\mathbf{Z}_p$ or $\mathbf{L}\,\mathbf{A} = -\mathbf{Z}_p$. Considering the power going into the coupled inductor, we

[4] Defining a NLTV coupled inductor with a TV, decrescent and positive definite energy function of the n state variables is a difficult task and is the subject of further research.

get:

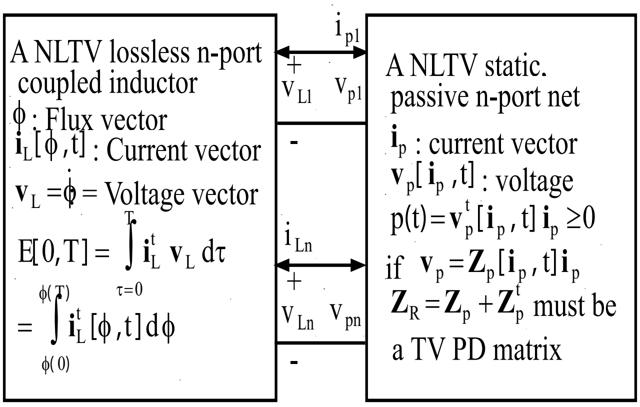

Figure 2: A general architecture for NLTV circuits. Constrain the elements to NLTI to prove the Kalman-Yakubovic LTI system stability lemma.

$$\begin{aligned} p_L(t) &= \mathbf{v}_L^t\,\mathbf{i}_L = -\mathbf{v}_p^t\,\mathbf{i}_p = -0.5(\mathbf{v}_p^t\,\mathbf{i}_p + \mathbf{i}_p^t\,\mathbf{v}_p) \\ &= -0.5(\mathbf{i}_p^t\,\mathbf{Z}_p^t\,\mathbf{i}_p + \mathbf{i}_p^t\,\mathbf{Z}_p\,\mathbf{i}_p) = -0.5\mathbf{i}_p^t\,(\mathbf{Z}_p^t + \mathbf{Z}_p)\mathbf{i}_p \\ &= -\mathbf{i}_p^t\,\mathbf{Z}_R\,\mathbf{i}_p = \mathbf{i}_p^t\,(\mathbf{L}\,\mathbf{A} + \mathbf{A}^t\,\mathbf{L})\mathbf{i}_p \end{aligned}$$

which will be negative when $\mathbf{Z}_R$ is positive definite. Then the solution of $\mathbf{L}\,\mathbf{A} + \mathbf{A}^t\,\mathbf{L} = -\mathbf{Z}_R$ should lead to an $\mathbf{L}$ matrix that is PD for any given PD matrix $\mathbf{Z}_R$, which is same as the K-Y lemma[5]. Note that for circuit synthesis corresponding to the LTI system $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$, we first fix the static, passive network, and test if the rest of the circuit turns out to be lossless or not.

The K-Y lemma and its interpretation using circuit equivalence shown here should be contrasted with testing for the stability of an equilibrium point of a nonlinear dynamics based on the Lyapunov first method, and its limitations. We first select the Lyapunov function candidate, which corresponds to first fixing the energy function or equivalently the reactive elements of a circuit realization. Then we look at the derivative along the system trajectory to see if it is negative definite, which corresponds to finding the rest of the circuit and checking if it is passive or not. We need to remind ourselves that

[5] From this circuit description, we can obtain a simple method for stability testing as follows: Using $\mathbf{L}\,\mathbf{A} = -\mathbf{Z}_p$ and assuming any non-symmetric matrix $\mathbf{Z}_p$, whose symmetric part $\mathbf{Z}_R$ is PD, solve for the matrix $\mathbf{L}$ (which in general will not be symmetric). Then we can conclude that the LTI system is stable iff $(\mathbf{L} + \mathbf{L}^t)$ is PD.

the Lyapunov approach does not lead to conclusive results all the time. The discussion above for LTI systems indicates that we need a different approach, where we will fix the passive circuit or the power consumption profile, and then look at the rest of the circuit.

Turning our attention to dynamics with chaotic behavior, we noted in the previous section that certain Type 3 circuits could lead to such dynamics. That is, we need to have at least one negative resistor to make the origin and any other finite valued equilibrium point unstable. Further, the circuit should behave as a passive circuit for large magnitudes of the state vector $\mathbf{x}$ so that the response remains bounded. We are all familiar with the Chen chaotic dynamics given by:

$$\begin{bmatrix} 0.2\,\dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -7 & 7 & 0 \\ -7 & 28 & -x \\ 0 & x & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{11}$$

which has a circuit realization as shown in Fig. 3. In this circuit realization, we have three LTI reactive elements, a NL gyrator and three resistors, one of which is negative. The negative resistor makes the origin an unstable equilibrium point. However, this particular circuit is not suitable to show why the other finite valued equilibrium point $\mathbf{x}_e^t = \begin{bmatrix} x_e & y_e & z_e \end{bmatrix} = \begin{bmatrix} \pm 3\sqrt{7} & \pm 3\sqrt{7} & 21 \end{bmatrix}$ is unstable or how the response remains bounded, and we need to find a different NLTI or NLTV circuit, taking us far into NLTV circuit synthesis, a research area still in its infancy. Note that circuit synthesis is similar to finding the proper Lyapunov function given the dynamics, and our inability to find one LF doesn't necessarily means that the equilibrium is not stable. Hence, from a practical design perspective, it is advantageous for us to form a Type 3 circuit with the required properties to arrive at the correct dynamics. We can arrive at one such NLTI circuit shown in Fig. 4, through simple modification to the circuit shown in Fig. 3. We have replaced the negative resistor by a nonlinear resistor with a voltage to current mapping, $i_R[v_R] = v_R(d + e * v_R^2)$, with d < 0, e > 0 and $|d| >> e$, which makes the resistor negative for small values of the voltage and passive for very large values. Further, we have made the NL gyrator matrix as a function of the state variable y(t) {rather than x(t)}, and changed the value of the capacitor connected to the first port. The first change implies that y(t) will be transmitted in any communication application, and the change is made so that global synchronization is possible. We will discuss this further when we discuss the synchronizing dynamics. We obtained chaotic response from this circuit for values of e in the range10^{-5} to 10^{-2} with d selected to place the equilibrium at the same point as Chen chaotic dynamics.

Again, using the circuit approach, a NLTV circuit can be obtained from the previous circuit by replacing all the LTI capacitors with LTV capacitors with a voltage mapping of the form, $v_c\,[q(t), t] = d_0\, q(t)\,(d_1 + d_2 * \sin^2(t))$

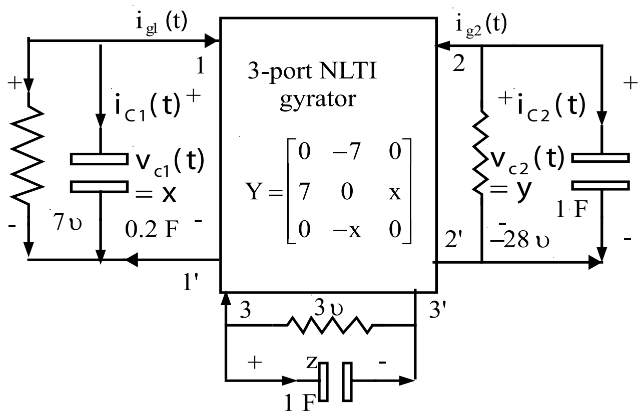

Figure 3: A circuit realization of Chen chaotic dynamics

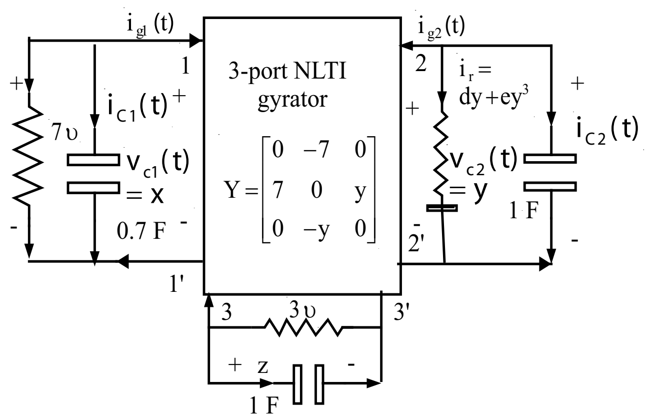

Figure 4: A new NLTI circuit with chaotic response (The dynamics can be made NLTV by replacing all the LTI capacitors by LTV or NLTV elements).

with all constants positive. The resulting dynamics is given by:

$$\begin{bmatrix} 0.7\,\dot{x}_t \\ \dot{y}_t \\ \dot{z}_t \end{bmatrix} = \begin{bmatrix} -7 & 7 & 0 \\ -7 & d + e * y_t^2 & -y_t \\ 0 & y_t & -3 \end{bmatrix} \begin{bmatrix} x_t\,(d_1 + d_2 * \sin^2(t)) \\ y_t\,(d_1 + d_2 * \sin^2(t)) \\ z_t\,(d_1 + d_2 * \sin^2(t)) \end{bmatrix} \quad (12)$$

Again, values in the range one for both d_1, d_2 with the equilibrium point same as that of Chen chaotic dynamics lead to chaotic response.

Let us now consider chaotic signal synchronization using the chaotic model in (12) where y_t is the signal being transmitted and available at the receiver, and using which we need to recover all the three state variables. Hence, we can note that this task is the same as that of state estimation from partial state observation, and hence the solution proposed here applies to that problem as well. If $\mathbf{x}_t^t = \begin{bmatrix} x_t & y_t & z_t \end{bmatrix}$, $\mathbf{x}_r^t = \begin{bmatrix} x_r & y_r & z_r \end{bmatrix}$ respectively, are the transmitter and the receiver state vectors, global synchronization requires that the error vector $\mathbf{e}^t = \begin{bmatrix} e_x & e_y & e_z \end{bmatrix} = \mathbf{x}_t^t - \mathbf{x}_r^t \rightarrow \mathbf{0}$ as $t \rightarrow \infty$ for all initial values of the error vector. To achieve this we need a receiver dynamics $\dot{\mathbf{x}}_r^t = f_r^t\,[\mathbf{x}_r\,,\, y_t\,,\, t]$, which when combined with the transmitter dynamics would lead to an error dynamics $\dot{\mathbf{e}}^t = f_e^t\,[\mathbf{e}\,,\, t] = f_e^t\,[\mathbf{x}_r\,,\, \mathbf{x}_t\,,\, t]$ with the origin as globally UAS. From an electrical circuits' perspective, this in turn implies that the error dynamics corresponds to that of a Type # 1 circuit defined above. Also, the error dynamics should be constrained such that the receiver dynamics obtained as $\dot{\mathbf{x}}_{\mathbf{r}}^t = \dot{\mathbf{x}}_t^t - \dot{\mathbf{e}}_t$ depends only on the receiver state variables and the transmitted state variable y_t and not the entire transmitter state vector. For transmitter dynamics in the form $\mathbf{L}\,\dot{\mathbf{x}}_t = \mathbf{A}[y_t\,,\, t]\,.\,\mathbf{x}_t$ where $\mathbf{L}$ is a PD matrix and $\mathbf{A}[y_t\,,\, t]$ is a NLTV matrix, this constraint requires that the error dynamics be of the form $\mathbf{L}\,\dot{\mathbf{e}} = \hat{\mathbf{A}}\,[\mathbf{x}_r\,,\, y_t\,,\, t]\,.\,\mathbf{e}$ where the columns of the NLTV matrix $\hat{\mathbf{A}}\,[\mathbf{x}_r\,,\, y_t\,,\, t]$ corresponding to the state variables not transmitted are same as the columns in $\mathbf{A}[y_t\,,\, t]$. Thus we have only the column corresponding to the transmitted variable to make the origin as the globally or locally UAS equilibrium point of the error dynamics. Look at the Chen chaotic dynamics in equation (11). When x_t is the transmitted variable, the value 28 in the (2,2) position of the $\mathbf{A}$ matrix makes it impossible to find such error dynamics. For the new NLTV chaotic dynamics given in (12) a family of error dynamics with the required properties can be written with little experience with our circuit approach as:

$$\begin{bmatrix} 0.7\,\dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} -7 & 7 & 0 \\ -7 & 0 & -y_t \\ 0 & y_t & -3 \end{bmatrix} \begin{bmatrix} e_x(d_1 + d_2 * \sin^2(t)) \\ e_y(d_1 + d_2 * \sin^2(t)) \\ e_z(d_1 + d_2 * \sin^2(t)) \end{bmatrix} - \begin{bmatrix} 0 \\ i_R\,[e_y, t] \\ 0 \end{bmatrix} \quad (13)$$

where $i_R\,[e_y\,,\, t]$ corresponds to the voltage to current mapping of a NLTV passive resistor. The circuit corresponding to this dynamics will be similar to Fig. 4 with only the resistor in port 2 different. We can easily prove that this dynamics has the origin as the globally UAS equilibrium point by forming a LF as the sum of energy in the three NLTV capacitors, and its derivative along the system trajectory as the negative of the sum of power consumed by the three resistors which are all passive and hence the derivative will be negative. We simulated the chaotic system and the synchronizing system using $i_R\left[e_y,t\right] = e_y + sign(e_y) * \left[|e_y|^{\frac{1}{2}} + {e_y}^2\right]$. The term ${e_y}^2$ dominates when $|e_y|$ is large and $|e_y|^{\frac{1}{2}}$ dominates when $|e_y|$ is very small. Synchronization happens in less than a second even when the initial errors are very large. We omit the simulation results to preserve space. A Matlab code is available for those who are interested.

5 Summary

A new approach for designing chaotic and their synchronizing systems is discussed. The method is very simple and easy to follow once we learn to look at NLTV systems as analogous circuits made of proper elements and not just some NLTV differential equations. The approach is applicable to a number of areas such as adaptive control and adaptive systems, feedback neural networks and fuzzy systems with memory. Some initial results on various applications can be found in ref.[5] and can be downloaded at no cost.

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email: journal@monotone.uwaterloo.ca
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