DESIGN OF STABLE SYMMETRIC AND NON-SYMMETRIC HALF-PLANE DIGITAL RECURSIVE FILTERS

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ABSTRACT

A method for the design of half-plane recursive filters is presented. The method is based on the construction of a 2V analog network with positive and negative reactive elements in one variable $s_1$, positive reactive elements in the other variable $s_2$, resistors and gyrators. The denominator $\Delta(s_1,s_2)$ of the driving point function of this network, which is shown to be mixed-min phase polynomial in $s_1$ and $s_2$ for all possible values of the network elements, is used along with double bilinear transformation to obtain stable half-plane recursive filter transfer functions. A design example for the case of non-symmetric half-plane filter is given.

I. INTRODUCTION

In the past few years, the design of two-dimensional (2D) digital filters has received wide attention. The interest is mainly due to the ever increasing availability of high performance/low priced digital components and a consequent shift towards digital signal processing techniques even where the original data are in analog form. Of the two types of 2D digital filters, progress in the design of recursive filters has been rather slow due to problems associated with stability and has been mainly restricted to the class of causal recursive filters.

Recently, Ekstrom and Woods [1] have demonstrated that 2D causal filters are capable of realizing a certain restricted class of magnitude functions. They showed that by relaxing the causality condition in one direction, a more general class of recursive filters, namely, non-symmetric half-plane (NSHP) recursive filters can be obtained. This class of filters can be used to realize the general class of positive definite magnitude functions. Using spectral factorization, they designed NSHP filters approximating a given magnitude response. However, their design procedure suffers from the fact that the filters obtained are in general unstable. Stabilization is achieved using window functions.

In this paper we consider the design of half-plane (symmetric and non-symmetric) recursive filters. The main motivation for this work stems from recent work by the authors in the design of causal 2D recursive filters [2], where they successfully applied the properties of two-variable (2V) passive networks to the design of 2D causal recursive filters that are guaranteed to be stable. In this paper, we adopt a similar approach and construct an analog network consisting of positive and negative reactive elements. The properties of such networks are derived, and it is shown that the numerator and the denominator of the driving point function of such networks can be used effectively to design half-plane filters that are guaranteed to be stable.

II. DESIGN PROCEDURE

In this section, we consider the design of stable two-dimensional half-plane (symmetric and non-symmetric) digital recursive filters. We restrict ourselves to linear shift-invariant filters whose transfer functions $H(z_1,z_2)$ can be represented as a ratio of two polynomials in $z_1$, $z_2$. Thus, for the case of symmetric half-plane (SHP) filters, we have

$$H_s(z_1,z_2) = \frac{N(z_1,z_2)}{D_s(z_1,z_2)}$$

and for the case of NSHP filters, $H(z_1,z_2)$ is given by

$$H_{ns}(z_1,z_2) = \frac{N_{ns}(z_1,z_2)}{D_{ns}(z_1,z_2)}$$

In designing filters with transfer functions as given above, we require that they be stable. A sufficient condition for stability is given by

$$D_s(z_1,z_2) \neq 0 \quad \text{for} \quad |z_1| = 1 \text{ and } |z_2| \leq 1 \quad (3)$$
for the case of SHP recursive filters and
\[ D_{ns}(z_1, z_2) \neq 0 \quad \text{for } |z_1| = 1 \text{ and } |z_2| \geq 1 \]  
(4a)
\[ D_{ns}(z_1, \infty) \neq 0 \quad \text{for } |z_1| \geq 1 \]  
(4b)
for the case of NSHP recursive filters. A two-variable polynomial satisfying the conditions in (3) and (4a) is generally referred to as mixed-min phase polynomial.

We can apply a double bilinear transformation given by
\[ z_i = \frac{1+s_i}{1-s_i^*} \quad i = 1, 2 \]  
(5)
to \( D_{ns}(z_1, z_2) \) and \( D_{ns}(z_1, \infty) \) to obtain the polynomials \( Q_6(s_1, s_2) \) and \( Q_3(s_1, s_2) \) respectively (here we neglect the factor \((1-s_i)^{n_1+n_2}(1-s_i^*)^2\) that comes in the denominators). The conditions in (3) and (4) then imply that, for symmetric filters,
\[ Q_6(s_1, s_2) \neq 0 \quad \text{for } \text{Re } s_1 = 0 \text{ and } \text{Re } s_2 \geq 0 \]  
(6)
and that, for non-symmetric filters,
\[ Q_6(s_1, s_2) \neq 0 \quad \text{for } \text{Re } s_1 = 0 \text{ and } \text{Re } s_2 \geq 0 \]  
(7a)
\[ Q_{ns}(s_1, 1) = (1-s_1)^n Q_1(s_1) \]  
(7b)
where \( Q_1(s_1) \) is a strictly-Hurwitz polynomial. That is
\[ Q_1(s_1) \neq 0 \quad \text{for } \text{Re } s_1 \geq 0. \]  
(8)
Therefore if we find a way to generate polynomials \( Q_6(s_1, s_2) \) and \( Q_{ns}(s_1, s_2) \) satisfying the conditions in (6) and (7a) respectively, we can apply the inverse bilinear transformation to obtain \( D(z_1, z_2) \) and thereby stable half-plane digital recursive filter transfer functions.

Now, consider the analog network as given in Fig. 1. The network consists of a lossless frequency-independent \((n_1+n_2+n_3+2)\) multiport which is terminated in \( n_1 \) ports with unit capacitance, \( n_1 \) ports with \(-1\) capacitance with unity \( s_2 \) capacitors and at one port with a unity resistor. Denoting the admittance matrix of the multiport as \( Y \), we obtain the general form of \( Y \) as
\[ Y = \begin{bmatrix}
0 & Y_{12} & Y_{13} & \cdots & Y_{1n} \\
-\frac{1}{s_2} & 0 & Y_{23} & \cdots & Y_{2n} \\
-\frac{1}{s_1} & -\frac{1}{s_2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{s_1} & -\frac{1}{s_2} & \cdots & \cdots & 0 \\
-\frac{1}{s_2} & 0 & \cdots & \cdots & 0 \\
\end{bmatrix} \]  
(9)
where \( y_{ij} \)'s are real constants. Writing the input admittance function at port 1 of the terminated multiport as
\[ Y_{in}(s_1, s_2, s_3) = P(s_1, s_2, s_3) \]  
(10)
where \( P(s_1, s_2) \) and \( \Delta(s_1, s_2) \) are two polynomials in \( s_1 \) and \( s_2 \), we can easily show that
\[ \Delta(s_1, s_2) = \text{det} [Y_{21} Y_{12}] \]  
(11)
\[ = \text{det} [\text{dia}(1-s_1 \cdots s_1, s_1 \cdots s_1, s_2 \cdots s_2) Y_{22}] \]
\[ = \begin{vmatrix}
1 & y_{23} & \cdots & y_{2k} & \cdots & y_{2m} & y_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
-\frac{1}{s_2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
-\frac{1}{s_1} & -\frac{1}{s_2} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-\frac{1}{s_2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{vmatrix} \]
\[ k = 3+n_1 \\
m = 3+n_1+n_2. \]  
(11)
We consider the properties of the polynomials \( P(s_1, s_2) \) and \( \Delta(s_1, s_2) \). The properties are given as Theorems I and II below:

Theorem I
The denominator \( \Delta(s_1, s_2) \) [and the numerator \( P(s_1, s_2) \)] of the driving point function of a 2V network consisting of positive and negative reactive elements in \( s_1 \), positive reactive elements in \( s_2 \), resistors and gyrators in a mixed-min phase polynomial in \( s_1, s_2 \) for all possible values of the network elements. That is, \( P(s_1, s_2) \) and \( \Delta(s_1, s_2) \) are two polynomials that satisfy the conditions given in (6) and (7a).

Proof
Replace all the negative capacitors in Fig. 1 by positive capacitors in \( s_2 \) domain to obtain a 3V passive network as shown in Fig. 2. Let the input admittance at port 1 of this network be written as
\[ Y_{in}(s_1, s_2, s_3) = \frac{P_1(s_1, s_2, s_3)}{Q_1(s_1, s_2, s_3)} \]  
(12)
From Figs. 1 and 2, it can be observed that
\[ Y_{in}(s_1,s_2) = Y'_{in}(s_1,s_2,s_3) \mid s_3 = s_1 \]  
(13)

and in particular
\[ \Delta(s_1,s_2) = Q'(s_1,s_2,s_3) \mid s_3 = s_1. \]  
(14)

\[ Y_{in}(s_1,s_2,s_3) \] is the driving point admittance function of a 2V passive network and hence it is positive real. Therefore, it immediately follows that \( Q'(s_1,s_2,s_3) \) is a strictly-Hurwitz polynomial [3]. That is
\[ Q'(s_1,s_2,s_3) \neq 0 \] for \( (\text{Re } s_1 \geq 0 \text{ and Re } s_2 \geq 0 \text{ and Re } s_3 \geq 0) \).  
(15)

Combining (14) and (15) we get
\[ \Delta(s_1,s_2) \neq 0 \] for \( (\text{Re } s_1 \geq 0 \text{ and Re } s_2 \geq 0 \text{ and Re } (-s_1) \geq 0) \)  
(16)

or
\[ \Delta(s_1,s_2) \neq 0 \] for \( (\text{Re } s_1 = 0 \text{ and Re } s_2 \geq 0) \).  
(17)

**Theorem II**

The polynomial \( \Delta_1(s_1) \) which is obtained from \( \Delta(s_1,s_2) \) as
\[ \Delta_1(s_1) = \Delta(s_1,s_2) \mid s_2 = 1 \]  
(18)

can be written as
\[ \Delta_1(s_1) = Q_h(s_1) Q_{nh}(s_2) \]  
(19)

where
\[ Q_h(s_1) = \sum_{i=0}^{n_1} q_{hi} s_1^i \neq 0 \] for \( \text{Re } s_1 \geq 0 \)  
(20a)

and
\[ Q_{nh}(s_1) = \sum_{i=0}^{n_2} q_{nih} s_1^i \neq 0 \] for \( \text{Re } s_1 \geq 0 \).  
(20b)

The proof of this theorem is quite simple and is omitted for brevity.

**Design of Stable Symmetric Half-Plane Filters**

As a result of Theorem I and from Fig. 1, we observe that we have a polynomial \( \Delta(s_1,s_2) \) which is mixed min phase. We also note that \( \Delta(s_1,s_2) \) remains mixed-min phase for all sets of values of \( Y_{ij} \)'s. We can therefore write the transfer functions \( H(s_1,s_2) \) of the SHP filter as
\[ H(s_1,s_2) = \frac{N_h(z_1,z_2)}{\Delta(s_1,s_2) (1+z_1)} \]  
(21)

The variables of optimization will be the coefficients \( y_{ij} \) and \( n_{ij}(k_1,k_2) \) and we can use any standard unconstrained optimization to obtain a transfer function that satisfies the given specifications.

**Design of Stable Non-Symmetric Half-Plane Filters**

The design procedure is again similar to the case of SHP filter design. We write the transfer function as
\[ H(s_1,s_2) = \frac{N_{ns}(z_1,z_2)}{\Delta(s_1,s_2) (1+z_1)} \]  
(22)

The coefficients \( n_{ns}(k_1,k_2) \) and \( y_{ij} \) are used as the variables of optimization. However, the coefficients \( y_{ij} \) have to be constrained such that
\[ \Delta(s_1,s_2) \mid s_2 = 1 \]  
(23)

which is possible because of Theorem II. It should be noted that the number of constraints is quite small and grows in proportion to \( n_1 \).

**III. NUMERICAL EXAMPLE**

Here we present some preliminary results on the application of the preceding method for the design of an NSHP recursive filter approximating a given magnitude response. The order of the transfer function is restricted to 2 in all directions \( (n_1=n_2=n_3=2) \). This transfer function is used to approximate the lowest filter specification:
\[ M(\omega_1,\omega_2) = 1 \mid |f_{2N}| < |f_{1N}| \]  
\[ = 0 \] otherwise  
(24)

where \( \omega_1 \) and \( \omega_2 \) are the normalized frequencies in directions 1 and 2, respectively. The transfer function is written as given in (22) and the coefficients \( n_{ns}(k_1,k_2) \) and \( y_{ij} \) form the variables of optimization. To reduce the number of variables optimization, we set \( y_{ij} = y_{2j} = y_{3j} = y_{3j} = 0 \), thus bringing the total number of variables of optimization to 28. We used a modified form of a constrained optimization technique developed by Haarhoff and Buys [4]. The result at the end of 100 iterations is given in Table I. In Fig. 3, a contour plot of the actual magnitude response is given. As can be seen from Fig. 3, we have obtained a reasonable approximation to the desired specifications.

**IV. CONCLUSIONS**

A method for the frequency domain design of stable half-plane digital recursive filters is presented. The method offers a complete solution for the design problem and is based on the construction of a 2V analog network with positive and negative reactive elements in one variable \( s_1 \), positive reactive elements in the other variable \( s_2 \), resistors and gyrators. It is shown that the numerator \( P(s_1,s_2) \) and the denominator \( \Delta(s_1,s_2) \) of
the driving point function of such a network are mixed-min phase polynomials in \(s_1, s_2\) \((i.e., P(s_1, s_2) \text{ and } \Delta(s_1, s_2) \neq 0 \text{ for } \text{Re } s_1 = 0 \text{ and } \text{Re } s_2 \geq 0)\) for all possible sets of values of the network elements. One of these mixed-min polynomials is assigned to the denominator of a 2V continuous function resulting in a stable half-plane (symmetric) continuous transfer function. The network elements are used as the variables of optimization thereby resulting in stable filters for all possible element values. The transfer function of a corresponding half-plane (symmetric) digital recursive filter is obtained from the continuous transfer function using bilinear transformation. Furthermore, it is shown that the transfer function of a stable non-symmetric half-plane digital recursive filter can be obtained from the continuous transfer function by using bilinear transformation and by suitably constraining the network element values. A simple example demonstrating the application of the method in designing 2D half-plane (non-symmetric) digital filters is given. Though we have not made any comparison with existing design procedures, it should be pointed out that this is the only method available so far for the design of half-plane recursive filters that are guaranteed to be stable.

REFERENCES


