where H^m is the symmetric tensor product of H with itself m times [8], and H^0 is just the real numbers. $E(H)$ may be regarded as a space of analytic nonlinear functionals on H by associating with $\Psi \in E(H)$ the functional

$$\Psi[\phi] = \langle \exp\phi, \Psi \rangle \tag{6}$$

where[1]

$$\exp\phi = \sum_{m=0}^{\infty} (m!)^{-1/2} \phi^m \tag{7}$$

and ϕ^m is the tensor product of ϕ with itself m times. The component of Ψ lying in H^m (denoted by $\Psi^{(m)}$) is just $(m!)^{1/2}$ times the mth degree Volterra kernel of (6).

Taking now $H' = E(H)$, a nonlinear analytic map from H' to H' may similarly be regarded as an element Λ of $H' \otimes E(H')$ via $\langle \Phi, \Lambda[\Psi] \rangle = \langle \Phi \otimes (\exp \Psi), \Lambda \rangle$ for all Φ and Ψ in H'. Our network kernels (2) and (3) are then $(i!)^{1/2}(m! j_1! \cdots j_m!)^{-1/2}[j_1, \cdots, j_m]$ times the component $\Lambda^{(i,j_1,\cdots,j_m)}$ of Λ lying in the

$$H^i \otimes (H^{j_1} \cup H^{j_2} \cup \cdots \cup H^{j_m})$$

subspace of $H' \otimes E(H')$, where $[j_1, \cdots, j_m]$ indicates the number of *distinct* ordering of the set $\{j_1, \ldots, j_m\}$, and $\cup$ indicates the symmetric tensor product in $E(H')$.

In practice, a Hilbert space is "too small" to include such useful possible Volterra kernels as those describing the case when *N is Z*. We need to replace the square-integrable kernels just described by generalized functions (distributions). This is possible if we replace H by a rigged Hilbert space [9] $K \subset H \subset K^*$, where K is a nuclear space and K^* is the dual space of generalized functions. It is straightforward to construct tensor products of rigged Hilbert spaces, and Dwyer [10], [11] has shown how the Fock space of a rigged Hilbert space (also a rigged Hilbert space) may be constructed. If we represent the map Z to N as an element of the space

$$E(K^*) \otimes E(E(K^*)) \tag{8}$$

we may choose our network kernels to be almost arbitrary generalized functions. Dwyer's work generalizes the special cases examined by Kristensen *et al.* [12] which constructed $E(K^*)$ for distributions on the real line, which work provided a rigorization for taking ordinary Volterra kernels to be distributions.

It goes without saying that by introducing suitable direct sums and tensor products of Hilbert or rigged Hilbert spaces, Volterra series for systems and networks involving arbitrary numbers of terminals, ports, inputs or outputs may be constructed by the above formalism. Also, the "component" Z may be replaced by any set of identical functional elements of a nonlinear system, each of which interacts with the system via two scalar functions of time, one of which is made dependent on the other in a time-invarient manner by the functional element.

[1] The factor $(m!)^{-1/2}$ replaces the expected factor $(m!)^{-1}$ in the expression for $\exp\phi$ because of our use (following references [8], [12]) of the inner product $\langle \phi^n, \psi^m \rangle_1 = \delta_{mn} \langle \phi, \psi \rangle^n$ in Fock space, rather than the more natural inner product $\langle \phi^n, \psi^m \rangle_2 = n! \delta_{mn} \langle \phi, \psi \rangle^n$. The Fock spaces $E(H)_1$ and $E(H)_2$ equipped with the $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ inner products, respectively, are naturally isomorphic, with ϕ^m in $E(H)_1$ associated with $(m!)^{-1/2}\phi^m$ in $E(H)_2$. Via this isomorphism, we have in $E(H)_2$ $\exp\phi = \Sigma(m!)^{-1}\phi^m$ as expected. In $E(H)_2$ we have $\exp(\phi+\psi) = (\exp\phi) \vee (\exp\psi)$, where $\vee$ is the symmetric tensor product, and in either $E(H)_1$ or $E(H)_2$, denoting the relevant inner product by $\langle\ ,\ \rangle$, we have $\langle \exp\phi, \exp\psi \rangle = \exp\langle \phi, \psi \rangle$. These properties justify the use of the expression $\exp\phi$ for the element given by (7) in $E(H)_1$. The use of the space $E(H)_2$ is generally simpler, and thus preferable, to $E(H)_1$, but we have used the latter since it would have been confusing to depart from the convention used in our references [8], [12].

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Maximally Flat Low-Pass Transfer Function Synthesis Using Continuous and Discrete Filters

L. T. BRUTON AND P. A. RAMAMOORTHY

***Abstract*—Discrete linear uniformly sampled digital filters are often employed in situations where there exists some preliminary linear analog continuous filtering operation. The latter continuous filter transfer function $P(s)$ is usually a low-pass function that is required for such purposes as reducing the effects of aliasing, limiting the dynamic range of the signal prior to digitization, etc. The realization of $P(s)$ is usually achieved quite independently of the realization of the subsequent discrete transfer function $H(z)$, where $z = e^{sT}$. In this contribution, it is shown that maximally flat solutions exist for the composite transfer function $P(s) \cdot H(z)$, so that it is possible to employ a simple baseband prefiliter $P(s)$ which, combined with $H(z)$, results in an overall maximally flat low-pass response.**

I. Introduction

The continuous linear transfer function $P(s)$ and the discrete linear transfer function $Q(z)$ are shown in Fig. 1(a) where the

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The authors are with the Department of Electrical Engineering, University of Calgary, Calgary, Alt., Canada.

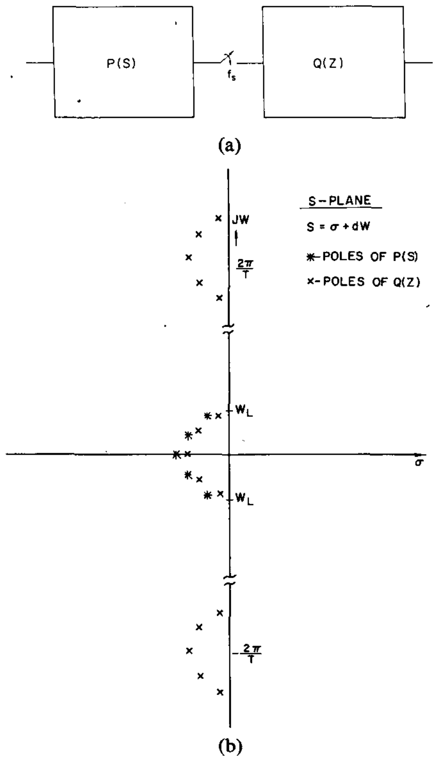

Fig. 1. (a) Composite filter structure. (b) Pole patterns for typical 10th-order composite filter.

composite transfer function $G(s,z)$ is simply given by

$$G(s,z)=P(s)\cdot Q(z). \tag{1}$$

In many practical situations, there exists some passband range $0<\omega<\omega_L$ over which a flat passband characteristic is desirable. That is, the function

$$|G(j\omega,e^{j\omega T})|=k_0+k(\omega)$$

where k_0 is a real constant and $k(\omega)$ is a real function of ω that is to be minimized over the passband region. The conventional procedure is to synthesize $P(s)$ and $Q(z)$ separately such that both functions have acceptable flat pass-band regions. For example, $P(s)$ might be realized as a Butterworth low-pass function and $Q(z)$ as an equiripple discrete low-pass function [1]. The typical singularity patterns for this conventional approach are sketched in Fig. 1(b) for a fifth-order low-pass function $P(s)$ in cascade with a fifth-order discrete low-pass function $Q(z)$. Of course, the periodicity of $Q(e^{j\omega T})$ ensures that the poles of $Q(z)$ repeat at integer frequency intervals of $2\pi/T$ along the direction of the $j\omega$-axis. Clearly the stopband response of the composite function $G(j\omega,e^{j\omega T})$ in the region of ω close to nonzero integer multiples of $2\pi/T$ is primarily determined by the stopband response of the function $P(j\omega)$. For example, at $\omega_S=2\pi/T$, the composite response is given by

$$|G(j\omega_S,e^{j\omega_S T})|=|P(j\omega_S)|\cdot|Q(1)|$$

and clearly this function can only be very much less than the zero-frequency passband response

$$|G(0,1)|=|P(0)|\cdot|Q(1)|$$

if

$$\left|\frac{P(j\omega_S)}{P(0)}\right|\ll 1.$$

II. Synthesis Procedure

The synthesis procedure is restricted to functions that do not have s-plane zeros. Thus, for an Mth-degree continuous filter

$$\frac{1}{P(s)}=\sum_{i=0}^{M}a_i s^i \qquad (\text{where } a_0=1) \tag{2}$$

and for an Nth-degree discrete filter

$$\frac{1}{Q(s)}=\sum_{i=0}^{N}c_i z^i. \tag{3}$$

From (1)–(3) it follows that we may write

$$|G(j\omega,e^{j\omega T})|^{-2}=\left(1+\sum_{i=1}^{M}B_i\omega^{2i}\right)\left(\sum_{i=0}^{N}D_i\cos\omega iT\right) \tag{4}$$

where the coefficients B_i, D_i are real and related to a_i, c_i, respectively, as given below

$$a_k^2+2(-1)^k\sum_{i=0}^{k-1}(-1)^i a_i a_{2k-i}=B_k, \qquad k=1,2,\cdots,M \tag{4.1}$$

$$\sum_{i=0}^{N}c_i^2=D_0 \tag{4.2}$$

$$2\sum_{i=0}^{N-k}c_i c_{i+k}=D_k, \qquad k=1,2,\cdots,N. \tag{4.3}$$

The maximally flat constraints are directly analogous to those employed in the conventional continuous Butterworth synthesis procedure except that they are applied to the mixed continuous discrete function in (4); that is, they are simply

$$|G(j\omega,e^{j\omega T})|^2_{\omega=0}-1=0 \tag{5}$$

and

$$\frac{d^k}{d\omega^k}|G(j\omega,e^{j\omega T})|^2=0, \qquad \text{at } \omega=0$$

$$\text{for } k=1,2,3,\cdots(2(M+N)-1). \tag{6}$$

From (4)–(6)

$$\sum_{i=0}^{N}D_i=1$$

$$B_j-B_{j-1}T^2\left(\sum_{i=1}^{N}D_i(i)^2\right)/2!+B_{j-2}T^4\left(\sum_{i=1}^{N}D_i(i)^4\right)/4!-\cdots$$

$$+(-1)^jT^{2j}\left(\sum_{i=1}^{N}D_i(i)^{2j}\right)/2j!=0,$$

$$\text{for } j=1,2,\cdots(M+N-1) \qquad 0 \qquad \text{for } i>M \tag{7}$$

and the passband cutoff frequency ω_L is simply defined by the additional constraining equation

$$|G(j\omega_L,e^{j\omega_L T})|^2=\frac{1}{2}. \tag{8}$$

Equation (7) with (8) are a set of $(M+N+1)$ nonlinear equations in $M+N+1$ unknown coefficients. The unique solution (for real coefficients) of this set of $(M+N+1)$ equations is obtainable by the Newton–Raphson algorithm [2]. Solutions for degree $M=1$, 2 and $N=3$, 4, 5 have been determined computationally with the above algorithm using a normalized upper cutoff frequency of 1 Hz and a sampling frequency of 20 Hz. The corresponding coefficients are given in Table I.

TABLE I

M	N	a_0	a_1	a_2	c_0	c_1	c_2	c_3	c_4	c_5	
2	3	1	1.62	1.0016	$-.22765\times10^3$	$.73897\times10^3$	$-.80883\times10^3$	$.29351\times10^3$			
1	4	1	1.0016		$.13828\times10^4$	$-.5998\times10^4$	$.97989\times10^4$	$-.71475\times10^4$	$.19647\times10^4$		
2	4	1	1.4162	1.0014	$.13289\times10^4$	$-.583\times10^4$	$.96162\times10^4$	$-.7071\times10^4$	$.19572\times10^4$		
2	5	1	1.805	1.0024	$-.84996\times10^4$	$.4595\times10^5$	$-.9967\times10^5$	$.10884\times10^5$	$-.59194\times10^5$	$.1297\times10^5$	

A Higher Degree Example

Consider a composite digital filtering process $G(s,z)$ that is to *approximate* the filtering characteristics of an eighth-order Butterworth continuous function. The response of this continuous Butterworth function is given as curve A in Fig. 2(a) and (b). The overall composite response $|G(j\omega, e^{j\omega T})|$ is assumed to have the specifications

passband cutoff frequency $=50$ Hz
sampling frequency $=1.25$ kHz

or in terms of ω_L and ω_S

$$\omega_L = 2\pi \times 50 \text{ rad/s} \tag{9}$$

$$\omega_S = 2\pi \times 1250 \text{ rad/s}. \tag{10}$$

Now, we *choose* to realize $P(s)$ as a simple second-order RC-active filter so that

$$M=2 \tag{11}$$

and therefore

$$N=6. \tag{12}$$

Equations (7)–(12) lead to the following nine $(2+6+1)$ simultaneous equations

$$\sum_{i=0}^{6} D_i = 1 \tag{13}$$

$$B_1 - T^2\left(\sum_{i=1}^{6} D_i(i)^2\right)/2! = 0 \tag{14}$$

$$B_2 - B_1 T^2\left[\sum_{i=1}^{6} D_i(i)^2\right]/2! + T^4\left[\sum_{i=1}^{6} D_i(i)^4\right]/4! = 0 \tag{15}$$

$$B_2 T^2\left[\sum_{i=1}^{6} D_i(i)^2\right]/2! - B_1 T^4\left[\sum_{i=1}^{6} D_i(i)^4\right]/4! + T^6\left[\sum_{i=1}^{6} D_i(i)^6\right]/6! = 0 \tag{16}$$

$$B_2 T^4\left[\sum_{i=1}^{6} D_i(i)^4\right]/4! - B_1 T^6\left[\sum_{i=1}^{6} D_i(i)^6\right]/6! + T^8\left[\sum_{i=1}^{6} D_i(i)^8\right]/8! = 0 \tag{17}$$

$$B_2 T^6\left[\sum_{i=1}^{6} D_i(i)^6\right]/6! - B_1 T^8\left[\sum_{i=1}^{6} D_i(i)^8\right]/8! + T^{10}\left[\sum_{i=1}^{6} D_i(i)^{10}\right]/10! = 0 \tag{18}$$

$$B_2 T^8\left[\sum_{i=1}^{6} D_i(i)^8\right]/8! - B_1 T^{10}\left[\sum_{i=1}^{6} D_i(i)^{10}\right]/10! + T^{12}\left[\sum_{i=1}^{6} D_i(i)^{12}\right]/12! = 0 \tag{19}$$

$$B_2 T^{10}\left[\sum_{i=1}^{6} D_i(i)^{10}\right]/10! - B_1 T^{12}\left[\sum_{i=1}^{6} D_i(i)^{12}\right]/12! + T^{14}\left[\sum_{i=1}^{6} D_i(i)^{14}\right]/14! = 0 \tag{20}$$

$$(1+B_1\omega^2+B_2\omega^4)\left(\sum_{i=0}^{6} D_i \cos \omega i T\right)\Bigg|_{\omega=\omega_L} = 2 \tag{21}$$

in the unknowns B_i, D_i. Solutions for these B_i, D_i coefficients, as obtained by the Newton–Raphson algorithm are given in the first row of Table II. The corresponding values of the polynomial coefficients a_i, c_i are obtained from B_i, D_i by using (4.1)–(4.3). The resultant values of a_i, b_i are the solution to the synthesis problem since, via (1)–(3), the transfer function $G(s,z)$ is determined. The a_i, b_i coefficients are given in Table II and a final realization is therefore given by

$$P(s) = 1/(1+a_1 s + a_2 s^2) \tag{22}$$

and

$$Q(z) = 1/(c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6). \tag{23}$$

The resultant magnitude function $|G(j\omega, e^{j\omega T})|$ is $|P(j\omega)| \cdot |Q(e^{j\omega T})|$ and is sketched as curve B in Fig. 2(a) and (b). Note that the linear graph, Fig. 2(a) does not reveal any significant difference between the conventional 8th-order Butterworth continuous response and the composite response $|G(j\omega, e^{j\omega T})|$. However on the logarithmic scale of Fig. 2(b) it is clear that in the region of $\omega \cong \omega_S$ the expected suppressed image is observed. This image is, in this case, greater than 50 dB below the passband value $|G(0,1)|$ and for many applications may be neglected. Of course, increasing ω_S will allow these images to be further suppressed.

The poles of $P(s)$ and $Q(e^{sT})$ may be calculated from (23) and (24) from which the Q of the pole-pair of $P(s)$ is found to be given by $Q=0.5907$. Therefore a simple single-amplifier RC-active filter circuit is sufficient for the circuit realization [3]. It is observed from Fig. 2(a) and (b) that the function $|G(j\omega, e^{j\omega T})|$ very closely approximates the passband and transition band responses of the 8th-order Butterworth continuous low-pass function and in the region of ω_S exhibits a stopband response that is determined by the rolloff characteristic of the second-order analog filter function $|P(j\omega)|$. The relationship between ω_S, ω_L and the magnitude A_S of the image at ω_S is given in Fig.

TABLE II

B_1	B_2	D_0	D_1	D_2	D_3	D_4	D_5	D_6
$.1888\times10^{-4}$	$.1042\times10^{-9}$	$.1447\times10^{11}$	$-.2483\times10^{11}$	$.1556\times10^{11}$	$-.6945\times10^{10}$	$.2096\times10^{10}$	$-.3841\times10^{9}$	$.3232\times10^{8}$
a_1	a_2	c_0	c_1	c_2	c_3	c_4	c_5	c_6
$.6269\times10^{-2}$	$.021\times10^{-4}$	$.1703\times10^{4}$	$-.1298\times10^{5}$	$.419\times10^{5}$	$-.7338\times10^{5}$	$.7373\times10^{5}$	$-.4024\times10^{5}$	$.9491\times10^{4}$

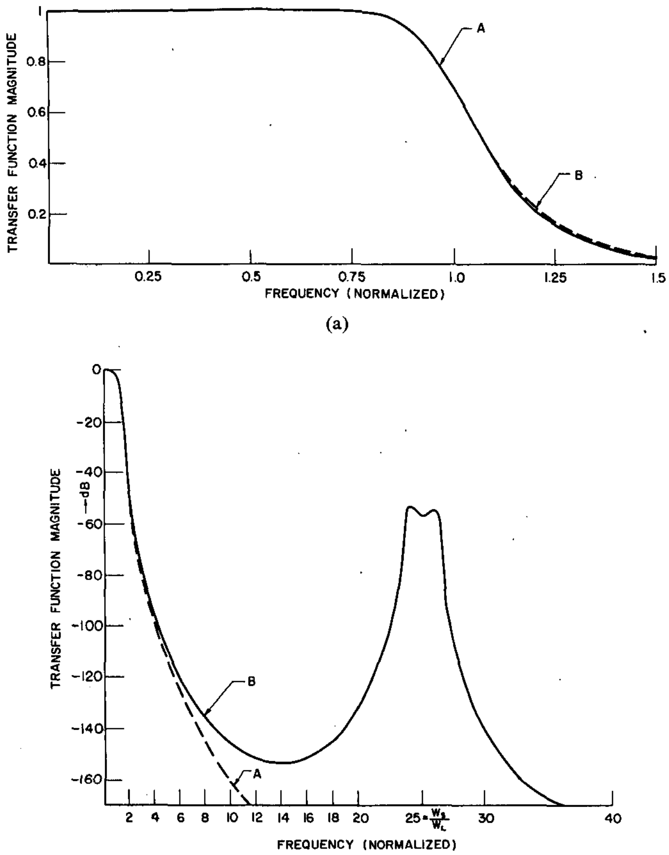

Fig. 2. (a) Passband response of 8th-order analog Butterworth filter and composite filter.

3, where this result is obtained by simply repeating the entire synthesis procedure for various ratios ω_S/ω_L.

The application of this type of synthesis procedure could be in the implementation of low-pass filters in pulse-code modulation (PCM) systems where a 3-kHz cutoff frequency and a 24-kHz

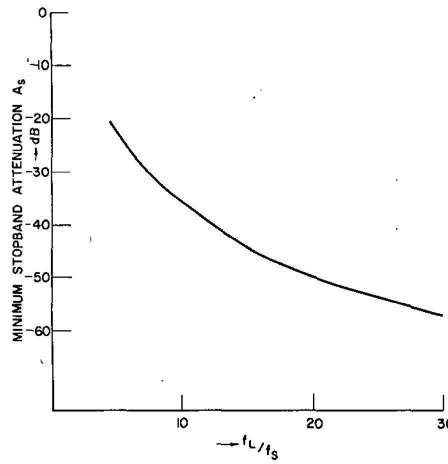

Fig. 3.

sampling frequency would yield a satisfactory rejection characteristics for $M=2$ and $N=6$. Furthermore, in applications that demand very *accurate matching* between parallel channels (cross-correlation measurements, for example) it is clear that the sixth-order ($N=6$) digital channels would be perfectly matched so that the only significant mismatch between channels would be attributable to a low-order ($M=2$) analog filter channel. This compares with the much greater difficulty that is involved in matching parallel high-order ($M=8$) analog filters as is required to obtain the same selectivity by direct analog implementation.

III. Conclusions

A method for the design of low-pass filters with the maximally flat passband criterion using a simple analog filter section in cascade with a higher order digital filter is presented. The method offers the advantage that, in applications involving high sampling rates ($\omega_s/\omega_L \gg 1$), it is possible to replace a high-order analog prefilter by a much simpler first- or second-order analog filter. The synthesis procedure is shown to be similar to that of the fully analog Butterworth synthesis and the solution of the synthesis equations is obtained numerically using the Newton–Raphson algorithm.

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