8. TIME-VARYING ELECTRICAL ELEMENTS, CIRCUITS, & THE DYNAMICS

8.1 Introduction to the Chapter

In the last chapter, we noted that some parameters associated with physical systems (that may or may not appear in the dynamical equations used to represent the systems) can vary with time (or forced to change) leading to time-varying or non-autonomous systems. We also found that, unfortunately, greater emphasis has been placed on the statement "vary with time" and the first statement "parameters associated with physical systems" is mostly ignored in the analytical approach to dealing with non-autonomous systems. In this chapter, we will consider how the time-varying characteristics can be introduced at the (electrical) element level and used to derive physically meaningful dynamics and practically useful results. Of course, the final results that we obtain are similar to the ones obtained using the analytical approach. However, the element or building block approach makes it easier to quantify the time varying behavior, arrive at meaningful expressions and limits on their variations, and to visualize their effect on the dynamics. Also, the element approach makes the distinction between non-autonomous linear dynamics and nonlinear autonomous systems very clear.

8.2 Time-Varying Electrical Elements

As we did in the case of linear and nonlinear time-invariant elements, we can use the basic properties of power consumption, generation, storage and return to define various time-varying elements. We also have to make sure that the time-varying behavior is properly associated with some physical properties or the structure of the device so that the basic properties of that device, for example power consumption, are maintained. Of course, if our interest is in more exotic devices, we can relax some of the constraints and study the resulting behavior.

8.2.1 Time-varying Resistors

Let us first consider the case of a resistor, a simple static device. In addition to the two variables, voltage and current, which are functions of time, we have seen, we can introduce additional variables to describe the effect of variation of some physical property of the structure of the device as a function of time. Thus, we may write:

\[ v(t, \alpha) = v[i(t), \alpha] \] (8.1)

for a current controlled resistor and as:

\[ i(t, \beta) = i[v(t), \beta] \] (8.2)

for a voltage controlled resistor. Here, \( \alpha \) and \( \beta \) are variables used to lump the effects of the various physical phenomena that are assumed to change as a function of time. The question is, how to constrain the above two general models so that they point to meaningful devices.

We can derive one simple model from (8.1) as:

\[ v(t, \alpha) = R[\alpha] i(t) \] (8.3a)

or

\[ i(t, \alpha) = \frac{1}{R[\alpha]} v(t) = G[\alpha] v(t) \] (8.3b)

where \( R[\alpha] \) (or \( G[\alpha] \)) represents the time varying resistance (conductance). The inverse of a time varying resistance (or conductance) may not exist as in the case of nonlinear resistors. Therefore, we will have to identify which is the independent variable and which is the dependent variable and use them accordingly. Some examples under this representation that can be derived from physical considerations are:

\[ R[\alpha] = R[T(t)] \] (8.4)

where \( R[T(t)] \) represents the variation of the resistance as a function of time. Or, we may have:

\[ \text{If we know what the various phenomena are and associate individual variables to each and every phenomena, we do not have to lump their effects into a single parameter. In such a case, we will have voltage-current relationships that include those parameters as well.} \]
where $L(t)$ represents the tap length of a rheostat (variable resistor) that may be varied as a function of time and $R_u$ represents the resistance for unit length, a positive quantity (Fig. 8.1).

We can make some important observations from the equations (8.3a, b). First, the model corresponds to a separable model. Second, as indicated before we need to clearly indicate the input and output variables and stick to that assignment. Finally, the two examples given are examples of linear time-varying resistors.

Considering the power consumed by the resistors described by (8.3), we obtain:

$$p[v, i, \alpha] = R[\alpha] i^2(t) = G[\alpha] v^2(t)$$  \hspace{1cm} (8.6)

Thus, we need to constrain $R[\alpha] = R[\alpha(t)] \{G[\alpha] = G[\alpha(t)]\}$ to be positive for all time if the resistor has to be passive (passive linear, time-varying resistors) for all the time. Such constraints can be easily included in the models derived from physical considerations such as the ones given in (8.4) and (8.5). In fact, in these equations we may include:

$$0 < T_{\min} \leq T(t) \leq T_{\max}$$  \hspace{1cm} (8.7)

and

$$0 < L_{\min} \leq L(t) \leq L_{\max}$$  \hspace{1cm} (8.8)

which in turn leads to:

$$0 < R_{\min} \leq R(t) \leq R_{\max}$$  \hspace{1cm} (8.9)

and

$$\int_t^{t+\tau} R(\tau)d\tau \geq \gamma = R_{\min}\tau$$  \hspace{1cm} (8.10)

This in turn implies, for a constant current,

$$0 \leq p_{\min} \leq p(t) \leq p_{\max}$$  \hspace{1cm} (8.11a)

and

$$E_R(\tau) = \int_t^{t+\tau} i^2 R(t)dt \geq i_R^2 R_{\min}\tau$$  \hspace{1cm} (8.11b)

That is, power will be consumed continuously (as long as the independent variable, current or voltage is non-zero) and a minimum amount of energy will be consumed by the resistor (and gets converted into heat) during any and every time interval $\tau$ for a given constant current or voltage. All of these properties have practical implications as we will see when we consider networks made of such elements.

An expression that fits the descriptions given above\(^2\) is:

$$R(t) = R_{\min}(1 + \alpha \sin^2[t]) \hspace{1cm} (8.12)$$

where $R_{\min}$ is the minimum value of the resistor and the maximum value is given by

$$R_{\max} = R_{\min}(1 + \alpha)$$  \hspace{1cm} (8.13)

Time varying expressions that have been considered by the classical control theorists and cast in the form of time-varying resistances are:

---

\(^2\) The condition of physical realizability is used to derive the general forms of time-varying expressions in the continuous domain. These equations can be used to build analog devices, to model analog systems, or implemented in the digital domain.
\begin{align*}
G_1(t) & = \frac{1}{(1 + t)^2} \quad (8.14) \\
G_2(t) & = \frac{1}{1 + t} \quad (8.15) \\
G_3(t) & = t \quad (8.16)
\end{align*}

Though these expressions are valid examples of time-varying functions, the connection to the physical elements is not generally made.

We can extend the model to include nonlinear time-variant elements by making the voltage-current relationship nonlinear when the other variable \( \alpha \) is fixed. An example of a \textit{passive nonlinear time-varying resistor} relationship is given by:

\[ v[i(t), \alpha] = R[\alpha] i^3(t) \quad (8.17) \]

where again \( R[\alpha] \) will be made a suitable positive function of the time to reflect the effect of the changes in physical parameters. A \textit{non-passive nonlinear time-varying resistor} will be of the form:

\[ v[i(t), \alpha] = R[\alpha] f[i(t)] \quad (8.18) \]

where \( f[i(t)] = 0 \) for \( i(t) = 0 \) and the characteristic \( f[i(t)] \) Vs \( i(t) \) can be in any of the four quadrants.

We do not have to restrict ourselves to \textit{separable functions} to describe a time-varying nonlinear resistor. We can choose non-separable function \( v[i(t), \alpha] \) of the two variables \( i(t) \) and \( \alpha \) which has the required properties. For example, for a passive resistor, \( v[i(t), \alpha] \) will have the following properties:

\begin{enumerate}
\item \( v[i(t) = 0, \alpha] = 0 \) regardless of the value of \( \alpha \) \quad (8.19a)
\item \( v[i(t), \alpha] > 0 \) for \( i(t) > 0 \) \quad (8.19b)
\item \( v[i(t), \alpha] < 0 \) for \( i(t) < 0 \) \quad (8.19c)
\end{enumerate}

For example, the expression:

\[ v[i(t), \alpha] = i(t) (i^2(t) + 2i(t) \sin(t) + 1) \quad (8.20) \]

meets the above requirements. In figure 8.2, we show the voltage Vs current characteristics (at different times) of this time-varying resistor. We can note that the waveforms are confined to first- and third-quadrants confirming that the time-varying resistor is indeed passive.

\textbf{8.2.2 Time-Varying Gyrators}

We can extend the concept of time-varying electrical elements to multi-port lossless memoryless devices. Let us consider the case of a \textit{two-port time-varying gyrator}. The I/O relationship for such a device will be of the form:

\[
\begin{bmatrix}
    i_1[t, \alpha] \\
    i_2[t, \alpha]
\end{bmatrix} =
\begin{bmatrix}
    0 & G[x, \alpha] \\
  -G[x, \alpha] & 0
\end{bmatrix}
\begin{bmatrix}
    v_1(t) \\
    v_2(t)
\end{bmatrix}
\quad (8.21)
\]

where we have assumed the port voltages as the independent variables and the port currents as the dependent variables and \( x \) is the state vector of the network.
in which the gyrator has been embedded. We can also use port currents as the independent variables to result in an I/O model:

\[
\begin{bmatrix}
  v_1(t) \\
  v_2(t)
\end{bmatrix} = \begin{bmatrix}
  0 & R(x, \beta) \\
  -R(x, \beta) & 0
\end{bmatrix} \begin{bmatrix}
  i_1(t) \\
  i_2(t)
\end{bmatrix}
\]  

(8.22)

In either case, we find that:

\[
p(t) = v_1(t)i_1(t) + v_2(t)i_2(t) = 0
\]  

(8.23)

indicating that a time-varying device described by (8.21) or (8.22) is lossless. Again, we can extend the definition to include multi-port (\(M>2\)) gyrators, define special classes and add constraints based on physical realizability. For example, for a two-port gyrator, a linear time-varying model will lead to:

\[
G(x, \alpha) = \alpha(t)\hat{G}[x] = \alpha(t)\hat{G} ; \quad \hat{G} > 0
\]  

(8.24)

where \(\alpha(t)\) will be defined based on the applications’ need. We can note that, unlike the case of the resistors, we can let \(\alpha(t)\) become positive, negative or zero. In the case of nonlinear time-varying 2-port gyrators, the admittance function will take the form:

\[
G(x, \alpha) = \alpha(t)\hat{G}[x]
\]  

(8.25)

Since we do not have any specific restrictions on the admittance (or impedance) matrix elements, we can use a general non-separable model if suitable functions can be chosen based on the problem being solved.

### 8.2.3 Time-Varying Transformers

We can also extend the definition for lossless transformers to nonlinear time-varying transformers. We omit the details as the process and the results will be very similar to the case of time-varying gyrators. In Table 8.1 we have provided some details and examples of time-varying transformers.

<table>
<thead>
<tr>
<th>Element</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonlinear, time-varying resistor (NLTV)</td>
<td>(v_0(t) = R(t)i_0(t)) for all (t) ; Separable, LTVC, (\infty &gt; R_{\max} &gt; R(0) &gt; R_{\min} &gt; 0) passive, current controlled.</td>
</tr>
<tr>
<td>NLTV charge controlled capacitor.</td>
<td>(v_c(t) = \frac{\delta E_c[\alpha(t)]}{\delta q} ), (p(t) = \frac{\delta E_c[\alpha(t)]}{\delta q} \frac{\delta q}{\delta t} + \frac{\delta E_c[\alpha(t)]}{\delta \alpha} \frac{\delta \alpha}{\delta t}) Power entering the cap. due to change in charge and due to the time-varying parameter.</td>
</tr>
<tr>
<td>NLTV flux controlled inductor.</td>
<td>(v_L(t) = 2k(\sin^2(t))q_{\alpha}^{2k} + \delta \alpha) Separable, TV, Nonlinear (E_L = \frac{1}{2}(q + f_1(t))^2 + f_2(t)^2 q_{\alpha}^{2k-1} (k=1), NLTV ) (\phi = 2k(\sin^2(t))q_{\alpha}^{2k} + \delta \phi) non-separable (+2(q + f_1(t))q_{\alpha}^{2k} ; f_1, f_2, q_{\alpha}^{2k}) bounded.</td>
</tr>
</tbody>
</table>

Table 8-1. Time-varying elements, their symbols, and the defining equations.
Let us now consider the case of time-varying elements with memory capacity. While considering physically realizable capacitors (which we represented earlier using time-invariant linear and nonlinear models), we observed that the charge \( q_c(t) \) on the plates of two parallel plates subjected to a voltage \( v_c(t) \) depends, in addition to the voltage, on the dielectric constant \( \varepsilon \) of the medium between the plates, the area \( A \) of the plates and the distance \( d \) between the plates. That is:

\[
q_c(t) = q_c[v_c(t), \varepsilon, A, d]
\]

(8.26a)

We modeled the effects of \( A \) and \( d \) to be linear and the effects of \( v_c(t) \) and \( \varepsilon \) to be linear or nonlinear to result in time-invariant linear and nonlinear models. We can look at some of these variables as potential candidates for obtaining time-varying behavior. It is obvious that we can visualize in particular varying the dielectric constant \( \varepsilon \) and or the distance between the plates \( d \) as a function of time subject to some constraints (Figure 8.3). Thus we can have:

\[
0 < \varepsilon_{\text{min}} \leq \varepsilon(t) \leq \varepsilon_{\text{max}}
\]

(8.26b)

\[
0 < d_{\text{min}} \leq d(t) \leq d_{\text{max}}
\]

(8.26c)

For modeling purposes, we can lump the effects of the change in these two parameters into a single parameter \( \alpha(t) \). For a voltage controlled capacitor, we can then write:

\[
q_c(t) = q_c[v_c(t), \alpha(t)]
\]

(8.27)
where $q_c[.]$ indicates the functional relationship. Similarly, for a charge controlled capacitor, we will have:

$$v_c(t) = v_c[q_c(t), \alpha(t)]$$  \hspace{1cm} (8.28)

We know losslessness and energy storage are the key properties of a capacitor (and an inductor). Thus, we can consider the energy $E_c[q_c, \alpha]$ stored in a time-varying capacitor and arrive at the following constraints (assuming $q_c = 0$ as the relaxation point):

$$E_c[q_c = 0, \alpha] = 0 \quad \text{for all values of } \alpha$$  \hspace{1cm} (8.29a)

$$E_c[q_c, \alpha] > 0 \quad \text{when } q_c \neq 0$$  \hspace{1cm} (8.29b)

and

$$E_L[q_c] \leq E_c[q_c, \alpha] \leq E_u[q_c]$$  \hspace{1cm} (8.29c)

where $E_L[q_c]$ and $E_u[q_c]$ are two positive definite functions. We arrive at the first constraint because no charge implies no stored energy regardless of what values $\varepsilon(t)$ and $d(t)$ take. This is based on the assumption that $q_c = 0$ is the relaxation point for our time-varying capacitor (Of course, we can modify the constraint to take care of the situation where the relaxation point is not the origin).

The second constraint follows from the definition for stored energy. The third equation simply states that the stored energy has to be bounded by some functions of the charge alone as the device represents a capacitor for any fixed value of $\alpha$.

We can take the differential of the energy with respect to time to obtain the power entering the capacitor as:

$$p(t) = \frac{dE_c[q_c, \alpha]}{dt} = \frac{\delta E_c[q_c, \alpha]}{\delta q} \frac{dq}{dt} + \frac{\delta E_c[q_c, \alpha]}{\delta \alpha} \frac{d\alpha}{dt}$$  \hspace{1cm} (8.30)

The power expression has two components; the first one is the power entering or delivered to the capacitor by the change in the charge and the second one corresponds to the power delivered by the mechanism that changes $\alpha$. We denote these two contributions as $p_q(t)$ and $p_\alpha(t)$. From the expression for $p_q(t)$ we get an expression for the voltage across the two plates of the capacitor as:

$$v_c(t) = \frac{\delta E_c[q_c, \alpha]}{\delta q}$$  \hspace{1cm} (8.31)

An example of a linear time-varying capacitor is given by the energy expression:

$$E_c[q(t), \alpha(t)] = \alpha(t)q^2(t)$$  \hspace{1cm} (8.32a)

The corresponding voltage expression is:

$$v_c(t) = \frac{\delta E_c[q, \alpha]}{\delta q}$$  \hspace{1cm} (8.32b)

That is, we have a separable model and the voltage depends linearly on the charge for a fixed value of $\alpha(t)$. The energy and voltage waveforms corresponding to this model are shown in figure 8.4. The corresponding expressions for an example of a nonlinear time-varying capacitor are:

$$E_c[q(t), \alpha(t)] = \alpha(t)q^4(t)$$  \hspace{1cm} (8.33a)

and

$$v_c(t) = \frac{\delta E_c[q, \alpha]}{\delta q}$$  \hspace{1cm} (8.33b)

An example of a non-separable model for nonlinear time-varying capacitors is:

$$E_c[q(t), \alpha(t)] = \frac{q^4(t)}{60} \left[ 10q^2(t) - 24q(t)\alpha(t) + 75\alpha^2(t) \right]$$  \hspace{1cm} (8.34a)

with

$$v_c(t) = q^3(t)\{q^2(t) - 2q(t)\alpha(t) + 5\alpha^2(t)\}$$  \hspace{1cm} (8.34b)
where \( \alpha(t) \) is a bounded time-varying function. The energy and voltage waveforms for such a time-varying capacitor when \( \alpha(t) = \exp(-t) \) are shown in figure 8.5.

Figure 8-4. a) Energy stored in and b) voltage across a time-varying linear capacitor as a function of time and charge.

Figure 8-5. a) Energy & b) voltage for a NLTV capacitor described by a non-separable model.
8.2.5 Time-Varying Inductors

A time-varying inductor, being the dual of a capacitor, can be defined in a similar manner. We simply summarize the results for the time-varying inductors in table 8.1.

8.3 Time-Varying Active and Non-passive Elements

As in the case of time-invariant nonlinear elements, we can constrain the dynamic elements as lossless elements and associate non-passive and or active behavior to resistive elements and power sources. Thus, we can have fully passive, fully active and non-passive time-varying resistors and use separable and non-separable, and linear and nonlinear models to describe such elements. We again omit the details as the derivations are straight forward.

8.4 Electrical Circuits with Time-Varying Elements & the resulting Dynamics

In this section, we consider some simple electrical circuits built using time-varying circuits and study the properties of such circuits. We will also make connections, where ever possible and or necessary, to theory developed using analytical approaches and point out the similarities, differences, and the superiority of the approach advanced in this book.

**Example 1** Let us consider a first-order circuit (Fig. 8.6) consisting of a linear time-invariant unit-valued capacitor and a linear time-varying resistor with a time-varying conductance given by:

$$G(t) = G_{\text{min}} \{1 + \alpha \sin^2(\beta t)\}$$

where $G_{\text{min}} > 0$ and $\alpha > -1$ for the resistor to be passive. The dynamics of the network is:

$$\dot{v}_c + G(t)v_c = \dot{v}_c + G_{\text{min}} \{1 + \alpha \sin^2(\beta t)\}v_c = 0$$

(8.36)

We can easily obtain a Lyapunov function (not just a candidate) for this dynamics by equating it to the energy left in the capacitor at any given time as:

$$\text{LF} = \frac{1}{2} v_c^2$$

(8.37)

The derivative of this LF along the system trajectory is:

$$\frac{d(\text{LF})}{dt} = v_c\dot{v}_c \geq G_{\text{min}} v_c^2$$

(8.38)

We can note that the LF derivative along the system trajectory is nothing but the negative of the power going into the time-varying resistor that is connected across the capacitor. In this example, the resistor is passive and power is consumed continuously. The power consumed is bounded as:

$$G_{\text{min}} (1 + \alpha) v_c^2 \geq P_R(t) = -\frac{d(\text{LF})}{dt} \geq G_{\text{min}} v_c^2$$

if $\alpha \geq 0$

(8.39)

$$G_{\text{min}} (1 + \alpha) v_c^2 \geq P_R(t) = -\frac{d(\text{LF})}{dt} \geq G_{\text{min}} (1 + \alpha) v_c^2$$

if $\alpha < 0$

That is, minimum power will be consumed by the resistor at any time until the capacitor voltage (and hence the stored energy) becomes zero. Thus, we can conclude that the dynamics given by equation (8.36) is absolutely stable. In Fig.
8.7, we show the various responses for two different initial conditions. As expected, all the variables go to zero as $t \to \infty$.

From this example, we should note that even though the dynamics is time-varying, the Lyapunov function need not necessarily be a time-varying function. Here, the derivative of the LF along the system trajectory is a time-varying one and the negative of the derivative is a decrescent function (dominated by a time-invariant positive definite function).

**Example 2**  Let us use the same network of Fig. 8.6 (for this example as well as for examples 3 and 4) and replace the conductance function by time-varying functions used in the control theory literature. First, we use:

$$G(t) = \frac{G_{\text{max}}}{1 + \alpha t}; \quad t \geq 0$$  \hspace{1cm} (8.40)

with $\alpha$ a positive constant. The conductance value which is always positive becomes smaller as time progresses and becomes exactly equal to zero as $t \to \infty$. Thus, its power absorption capacity decreases as time progresses. The dynamics corresponding to this resistance is:

$$\dot{v}_c + G(t)v_c = \dot{v}_c + \frac{G_{\text{max}}}{1 + \alpha t} v_c = 0$$  \hspace{1cm} (8.41)

Since the reactive element is the same, we have the same Lyapunov function:

$$LF = \frac{1}{2} v_c^2$$  \hspace{1cm} (8.42)

However, the LF derivative along the system trajectory changes to:

$$\frac{d(LF)}{dt}_{\text{sys. trajec.}} = v_c \dot{v}_c = -\frac{G_{\text{max}}}{1 + \alpha t} v_c^2$$

and is bounded by:

$$-G_{\text{max}} v_c^2 \leq \frac{d(LF)}{dt}_{\text{sys. trajec.}} \leq 0$$  \hspace{1cm} (8.44)

Again, the derivative is a decrescent function and becomes zero as and when the capacitor voltage becomes zero and stays there. Thus, we can conclude that the dynamics based on the resistor in equation (8.40) will also be absolutely stable. In Fig. 8.8, we show the various responses for two different initial conditions. Again, as expected, all the responses go to zero as $t \to \infty$.

**Example 3**  The conductance expression used for this example is given by:

$$G(t) = \frac{G_{\text{max}}}{(1 + \alpha t)^2}; \quad t \geq 0$$  \hspace{1cm} (8.45)

We can observe that the conductance value at any time-instance is positive for positive $\alpha$. Also, the conductance tends to zero much faster as time progresses as compared to the time-varying conductance of the previous example. In fact, we can show that the infinite integral:
\[ \int_{\tau=0}^{\infty} G(\tau) d\tau \]  
\( (8.46) \)

is finite for the conductance of this example where as it is infinite for the conductance of the previous example.

The dynamics based on this conductance is given by:

\[ \dot{v}_c + G(t)v_c = \dot{v}_c + \frac{G_{\text{max}}}{(1 + \alpha t)^2} v_c = 0 \]

\( (8.47) \)

Again, the LF is the same given by the energy left in the linear capacitor:

\[ \text{LF} = \frac{1}{2} v_c^2 \]

\( (8.48) \)

and the LF derivative along the system trajectory is:

\[ \left. \frac{d(LF)}{dt}_{\text{sys. trajec.}} \right|_{\text{sys. trajec.}} = v_c \dot{v}_c \]

\( (8.49) \)

or

\[ -G_{\text{max}} v_c^2 \leq \left. \frac{d(LF)}{dt}_{\text{sys. trajec.}} \right|_{\text{sys. trajec.}} \leq 0 \]

\( (8.50) \)

Thus, though the derivative is negative, it can (and does) become zero before the capacitor voltage becomes exactly equal to zero. Thus, the dynamics resulting from the use of the resistor of equation (8.45) is stable, but not absolutely stable. For absolute stability, the integral of the conductance function given by (8.46) has to be infinite. A proof of this requirement, though simple to derive, is not really necessary, and hence is omitted. In Fig. 8.9, we show the various responses for two different initial conditions. Note that though the current in the circuit becomes zero as \( t \to \infty \), the voltage across the LTI capacitor and hence the energy left in the capacitor does not become zero.

**Example 4** Finally, we consider the same first order network of Fig. 8.6. using a conductance expression of the form:

\[ G(t) = \alpha \cdot (1 + \alpha t), \quad \alpha \geq 0 \]

\( (8.51) \)

Here we have the diametrically opposite situation. The conductance value increases (linearly) as the time increases, and hence its appetite for power.

The dynamics, the LF, and its derivative along the system trajectory using this conductance are:

\[ \dot{v}_c + G(t) v_c = \dot{v}_c + \alpha v_c \]

\( (8.52a) \)

\[ \text{LF} = \frac{1}{2} v_c^2 \]

\( (8.52b) \)

\[ \left. \frac{d(LF)}{dt}_{\text{sys. trajec.}} \right|_{\text{sys. trajec.}} = v_c \dot{v}_c \]

\( (8.52c) \)

\[ -\alpha v_c^2 \leq \left. \frac{d(LF)}{dt}_{\text{sys. trajec.}} \right|_{\text{sys. trajec.}} \leq 0 \]

\( (8.52d) \)

Figure 8-8. Response of the time-varying network of figure 8.6, when the time-varying conductance is changed to \( G(t) = G_{\text{max}}/(1 + \alpha t) \), \( \alpha \) a positive constant. a) Voltage across the capacitor; b) Energy left in the capacitor; c) Current through the time-varying resistor, and d) Power consumed by the resistor.
or

\[-\beta \leq \frac{d(LF)}{dt}_{\text{sys. trajec.}} \leq 0 \quad \text{where} \quad \beta > 0 \quad (8.52d)\]

and

\[\frac{d(LF)}{dt}_{\text{sys. trajec.}} = 0 \quad \text{only when} \quad v_c = 0 \quad (3.52e)\]

Thus, we can conclude that the dynamics will be at least absolutely stable. From the observation that the conductance and hence the power absorption capacity increases as time goes by, we can speculate that the dynamics will be exponentially stable. In fact, the condition:

\[
\int_{t}^{\infty} G(\tau) d\tau \geq \gamma > 0 \quad (8.53)
\]

that we encountered before (using the resistance definition) implies exponential stability, and is satisfied by the resistor used in this example. In Fig. 8.10, we show the response of the dynamics 8.52 for some initial conditions. As discussed, the response goes to zero. Also by comparing figures 8.7 to 8.9 with fig. 8.10, we find that the response of dynamics of example 1 and 4 go to zero faster, whereas the response of example 2 and 3 are somewhat sluggish. In fact, for some initial conditions, the response of example 3 settles at some non-zero value as indicated earlier.

Figure 8-9. Response of the time-varying network of figure 8.6. when the time-varying conductance is changed to \( G(t) = G_{\max}/(1+\alpha t)^2 \), \( \alpha \) a positive constant. a) Voltage and b) Energy left in the capacitor; c) Current through and d) Power consumed by the resistor.

Figure 8-10. Response of the time-varying network of figure 8.6. when the time-varying conductance is changed to \( G(t) = \alpha t \), \( \alpha \) a positive constant. a) Voltage and b) Energy left in the capacitor; c) Current through and d) Power consumed by the resistor.
Examples 5 & 6  Now, we consider two examples corresponding to second order dynamics. Both the examples use a linear time-invariant capacitor, a linear time-invariant inductor and one or more time-varying linear resistors. The network corresponding to the first dynamics is shown in Fig. 8.11. The inductor value is set at one Henry and the capacitance value at 0.2 Farad. The time-varying resistance is given by:

$$R(t) = 1 + t$$  \hspace{1cm} (8.54)

where the resistance increases linearly with respect to the time variable. The dynamics of the network is:

$$\begin{bmatrix}
\frac{1}{5} \dot{v}_c(t) \\
\dot{i}_L(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & -(1+t)
\end{bmatrix}
\begin{bmatrix}
v_c(t) \\
i_L(t)
\end{bmatrix}$$  \hspace{1cm} (8.55)

which is linear and time-variant. We find that $$\begin{bmatrix} 0 & 1 \end{bmatrix}$$ is the only equilibrium point for this dynamics. We can write the LF as the sum of the energy left in the two linear reactive elements as:

$$\text{LF} = \frac{1}{2} \left( \frac{1}{5} v_c^2(t) + i_L^2(t) \right)$$  \hspace{1cm} (8.56)

which is time-invariant (depends only on the state vector and not the time variable explicitly). The derivative of the LF along the system trajectory is:

$$\frac{d(\text{LF})}{dt} = \begin{bmatrix} v_c(t) & i_L(t) \end{bmatrix}
\begin{bmatrix}
\frac{1}{5} \dot{v}_c(t) \\
\dot{i}_L(t)
\end{bmatrix}$$  \hspace{1cm} (8.57)

As noted above, the derivative can become zero as soon as one state variable, the current through the inductor, becomes zero and hence the dynamics may not reach the equilibrium point ($$v_c \neq 0$$). However, from the condition $$i_L = 0$$, $$v_c \neq 0$$ and the network dynamics we find that $$i_L$$ and hence $$v_c$$ will change. The passive resistance present in the circuit which keeps consuming energy as long as the current through is not zero will then push the state variables close, and finally to the origin, the equilibrium point. Hence, we can conclude that the dynamics is absolutely stable. In Fig. 8.12, we show the various responses for two initial conditions. The two state variables, the voltage across the capacitor, and the current through the inductor, are shown in figures 8.12a and 8.12b respectively. The voltage across the resistor appears in figure 8.12c. The energy left in the circuit and the power consumed by the resistor are shown in Fig. 8.12d. A Phase plane plot using the state variables of the network appears in figure 8.12e. A Phase plane plot using the energy in the capacitor and the energy in the inductor is shown in figure 8.12f. We can note that the state goes to the origin, the equilibrium point, as expected. Also, we can see from figure 8.12e that the inductor current becomes zero making the derivative of the LF momentarily zero. However, the dynamics moves the state away from that point such that the total stored energy gets depleted completely.

In this example, we used a resistor whose power absorption depends on one state variable (current through the inductor) which in turn is equal to the derivative of the second state variable (voltage across the capacitor) because of the way the elements are connected. Also, as we observed earlier, the resistance value goes to infinity in a linear fashion. A faster move to infinity (which implies large power consumption if the current is not zero) may make the current itself zero (zero power consumption) because of the network structure (or equivalently the dynamics) while the other state variable, the voltage across the capacitor, may not become zero. That is, we can end up with a dynamics whose equilibrium point is not absolutely stable. We will illustrate this in the next example.
Example 6  We use the same network of Fig. 8.11 for this example, but change the inductor value to one and the resistor characteristic to:

\[ R(t) = \alpha_1 + \alpha_2 e^{\beta t} \]  \hspace{2cm} (8.58)

with \( \alpha_1 \) and \( \alpha_2 > 0 \) (resistor becomes passive) and \( \beta > 0 \) makes the resistance grow exponentially. The corresponding dynamics, Lyapunov function, and its derivative along the system trajectory are:

\[
\begin{bmatrix}
\dot{v}_c(t) \\
\dot{i}_L(t)
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
-1 & -(\alpha_1 + \alpha_2 e^{\beta t})
\end{bmatrix}
\begin{bmatrix}
v_c(t) \\
i_L(t)
\end{bmatrix}
\]  \hspace{2cm} (8.59a)

\[
LF = \frac{1}{2} \left( v_c^2(t) + i_L^2(t) \right)
\]  \hspace{2cm} (8.59b)

\[
\frac{d(LF)}{dt}_{\text{sys trajec.}} = \begin{bmatrix}
v_c(t) & i_L(t)
\end{bmatrix}
\begin{bmatrix}
\dot{v}_c(t) \\
\dot{i}_L(t)
\end{bmatrix}_{\text{sys trajec.}}
= \begin{bmatrix}
v_c(t) & i_L(t)
\end{bmatrix}
\begin{bmatrix}
0 \\
-1 & -(\alpha_1 + \alpha_2 e^{\beta t})
\end{bmatrix}
\begin{bmatrix}
v_c(t) \\
i_L(t)
\end{bmatrix}
= -(\alpha_1 + \alpha_2 e^{\beta t})i_L^2(t)
= 0 \text{ for } i_L = 0 \text{ and any value for } v_c
\]  \hspace{2cm} (8.59c)

Again, we note that the derivative of the Lyapunov function can become zero when one state variable becomes zero regardless of the value of the other state variable. The fact that the resistance value moves to infinity much faster (in an exponential manner) makes the situation much worse. From a network perspective, this implies that the resistor gets removed (or becomes an open circuit) before all the energy in the capacitor is dissipated. Therefore, the state will not reach the equilibrium point, and hence the equilibrium point is not stable. In fact, for any \( v_c(0) \) and \( i_L(0) = -(\alpha_1/(\alpha_1 + \alpha_2))v_c(0) \), we can obtain a closed form solution for the dynamics as:

\[
v_c(t) = \frac{\alpha_1 v_c(0)}{\alpha_1} \left( \frac{\alpha_2}{\alpha_1} + e^{-\alpha_1 t} \right)
\]  \hspace{2cm} (8.60)

\[
i_L(t) = \frac{\alpha_1 v_c(0)}{\alpha_1 + \alpha_2} e^{-\alpha_1 t} \hspace{2cm} t \geq 0
\]

implying that \( v_c(t) \to \alpha_2 v_c(0)/(\alpha_1 + \alpha_2) \) and \( i_L(t) \to 0 \) as \( t \to \infty \). In Fig. 8.13 we show the results of simulation for some initial conditions that are connected as discussed above. We find that the voltage across the capacitance settles at some non-zero value as expected.
Referring back to the resistor expression, we find that in effect we have two resistors in series, one LTI and passive with a value of $\alpha_1$ ohms and the other time-varying and passive with a value given by $\alpha_2 e^{\beta t}$. In ordinary circumstance, the presence of the LTI passive resistor alone would have made the equilibrium point absolutely stable. However, the series interconnection and the fact that the second resistor becomes an open circuit as time progresses makes the presence of the other LTI resistor irrelevant. The building block approach and the network perspective can help overcome such problems. In this particular case, the stability of the equilibrium point can be achieved by completely removing that resistor (the network becomes LTI) or by making the time-varying resistor value bounded.

**Example 7** In this example we consider a slightly more complex network consisting of two linear time-invariant capacitors and a time-varying two-port network as shown in Fig. 8.14a. The two-port network can be represented by a time-varying two-port gyrator and a number of time-varying passive resistors as shown leading to the architecture shown in Fig. 8.14b. The network therefore is passive. The dynamics of this network is:

$$
\begin{bmatrix}
\dot{v}_{c1}(t) \\
\dot{v}_{c2}(t)
\end{bmatrix} =
\begin{bmatrix}
-1 & -e^{-2t} \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
v_{c1}(t) \\
v_{c2}(t)
\end{bmatrix}
$$

(8.61)

The dynamics has only one equilibrium point, the origin. Since the storage elements are time-invariant, we can still write the Lyapunov function as the energy left in the storage elements:

$$
LF = \frac{1}{2} \left( v_{c1}^2(t) + v_{c2}^2(t) \right)
$$

(8.62)

and the derivative of the LF along the system trajectory is:

$$
\frac{d(LF)}{dt}_{\text{sys. trajec.}} =
\begin{bmatrix}
v_{c1}(t) & v_{c2}(t)
\end{bmatrix}
\begin{bmatrix}
\dot{v}_{c1}(t) \\
\dot{v}_{c2}(t)
\end{bmatrix}_{\text{sys. trajec.}}
$$

$$
= [v_{c1}(t) \ v_{c2}(t)]
\begin{bmatrix}
-1 & -e^{-2t} \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
v_{c1}(t) \\
v_{c2}(t)
\end{bmatrix}
$$

$$
= -(v_{c1}^2(t) + v_{c2}^2(t)) + v_{c1}(t)v_{c2}(t)(1 - e^{-2t})
$$

$$
= -\left((v_{c1}(t) + \alpha v_{c2}(t))^2 + (v_{c2}(t) \sqrt{1-\alpha^2})^2\right) \quad \text{where} \quad 0 \leq \alpha = \frac{1-e^{-2t}}{2} \leq \frac{1}{2}
$$

(8.63)

Figure 8-13. Response of the time-varying network of figure 8.11. with the time-varying resistance as $R(t) = \alpha_1 + \alpha_2 e^{\beta t}$ where $\alpha_1$, $\alpha_2$, and $\beta$ are positive constants. a) The voltage across the capacitor; b) Current through the inductor; c) The voltage across the resistor; d) Energy left in the circuit and the power consumed by the resistor; e) Phase plane plot.
which remains negative as long as any of the state variable is non-zero and is a decrescent function. Thus, we can conclude that the dynamics in (8.61) is absolutely stable. In figure 8.15, we show the various responses for some initial conditions. We can note that the state value reaches the origin as expected.

Figure 8-14. a) A network with two LTI capacitors and a time-varying two-port network; b) An alternate realization of the same dynamics using time-varying resistors and a time-varying gyrator.

Figure 8-15. The response of the network of figure 8.14: a) & b) The voltage across the two LTI capacitors; c) The phase plane trajectory; d) The phase plane trajectory in terms of the energy left in the two capacitors.
Example 8  In this example, we consider a first-order network with a charge controlled linear time-varying capacitor and a linear time-invariant resistor of value $G$ mhos as shown in figure 8.16. The capacitance is described by the stored energy expression:

$$E_c[q,t] = \frac{1}{2} \{1 + \alpha \sin^2(\beta t)\} q^2(t)$$ \hspace{1cm} (8.64)

with $\alpha > -1$, and the voltage expression:

$$v_c(t) = \frac{\delta E_c[q,t]}{\delta q} = (1 + \alpha \sin^2(\beta t)) q(t)$$ \hspace{1cm} (8.65)

The dynamics of the network is given by:

$$\dot{q} + G v_c = \dot{q} + G(1 + \alpha \sin^2(\beta t)) q(t) = 0$$ \hspace{1cm} (8.66)

which is same as the dynamics of a network consisting of a linear capacitor and a time-varying resistor as given by example 1. However, the true Lyapunov function, the stored energy in the reactive element for this network, is different and is given by:

$$LF = E_c = \frac{1}{2} \{1 + \alpha \sin^2(\beta t)\} q^2(t)$$ \hspace{1cm} (8.67)

which is a decrescent function. The derivative of this function along the system trajectory is:

$$\frac{d(LF)}{dt} = \frac{dE_c}{dt} = \left\{ q^2(t) \alpha \beta \sin(\beta t) \cos(\beta t) + (1+\alpha \sin^2(\beta t)) q(t) \dot{q}(t) \right\}_{\text{sys. trajec.}}$$

$$= \frac{1}{2} q^2(\alpha \beta \sin(2\beta t) - G(1 + \alpha \sin^2(\beta t))^2 q^2(t) - 0.5 \alpha \beta \sin(2\beta t) q(t) \dot{q}(t))_{\text{sys. trajec.}}$$

$$= \frac{1}{2} q^2(1 + \alpha \sin^2(\beta t))^2 - 0.5 \alpha \beta \sin(2\beta t) q(t) \dot{q}(t)_{\text{sys. trajec.}}$$

$$\leq 0 \text{ if } G \geq \begin{cases} 0.5[\alpha \beta] \quad \text{when } -1 < \alpha < 0 \\ 0.5[\alpha \beta]^2 \quad \text{when } \alpha \geq 0 \end{cases}$$ \hspace{1cm} (8.68)

which is also different from the one seen in example 1. From the above expression, we find a sufficiency condition for absolute stability is that the conductance value must be greater than some minimum value. This will ensure that the extra energy which is being added by the time-varying mechanism will also be consumed in a timely manner so as to bring the state to the equilibrium value. Note that this condition is different from what we obtained in example 1 which simply states that the conductance has to be positive.

We show in Fig. 8.17 simulation results for both examples for two values of $G$. We show not only the state trajectory (which is same for both) but also other variables such as energy left etc. (which are different). In figures 8.17 a to c, we show the results when $\alpha = 3, \beta = 2\pi$ and $G = 20$. From the above expression we find that these values lead to a derivative of the Lyapunov function that is always negative implying that the dynamics is stable and the simulation results confirm the same. In figures 8.17 d to f, we show the results of simulation with $G = 0.25$. For this value of $G$, the derivative of the Lyapunov function becomes both positive and negative (as can be seen from the above expression as well as figure f) implying that we cannot say if the equilibrium is stable based on this Lyapunov function. However, the simulation results (as well the equivalence to example 1 and the resulting condition that the conductance only needs to be positive) indicate that the dynamics is stable for this small value of $G$ as well.

This example illustrates the key aspect of stability analysis using Lyapunov method. It is only a sufficiency condition and not a necessary one. From a network perspective, Lyapunov condition amounts to saying that a dynamics is stable if the net power going into the energy storage devices is always negative. And as this example demonstrates, we can have situations where this need not necessarily be true, but the dynamics can still be stable. In design, where we invariably use conservative figures, we may use element values that obey the Lyapunov condition. This is especially true for higher order systems where we may not be able to arrive at alternate Lyapunov functions (as we did for this first order example) which may point to different element values. We will illustrate this in example 10.
Figure 8-17. The response of the first-order time-varying network of figure 8.16 with $\alpha = 3$, $\beta = 2\pi$ and $G = 20$. a) The charge and the voltage across the time-varying capacitor; b) The energy left in the time-varying capacitor; c) The negative of the derivative of the Lyapunov function ($=-p_c(t)$, net power leaving the capacitor). We also show the response from a net with a LTI unit valued capacitor & a time-varying resistor that leads to the same dynamics (example 1). In figure b, we have the energy ($=0.5q^2(t)$), and in figure c, we show the power consumed by the time-varying resistor ($p_R(t)$). We can see that though the state is the same for both dynamics, the rest of the responses are not.

Fig. 8.17 Contd. The response of the first-order time-varying network of figure 8.11 with $\alpha = 3$, $\beta = 2\pi$ and $G = 0.25$. d) The charge and the voltage across the time-varying capacitor; e) The energy left in the time-varying capacitor; f) The negative of the derivative of the Lyapunov function ($=-p_c(t)$, net power leaving the capacitor). In this case, we can observe that the net power leaving the capacitor becomes negative implying that at times the power consumed by the LTI resistor is less than the power injected by the time-varying phenomena. However, eventually, the state variable goes to zero indicating that the dynamics is still absolutely stable.
Example 9  In this example we consider the dynamics obtained from a network consisting of a charge controlled nonlinear time-varying capacitor and a linear resistor of conductance G mhos (Fig. 8.18a). The capacitance is described by the expression for stored energy:

$$E_c[q(t), \alpha(t)] = \alpha(t)q^4(t)$$

$$= \frac{1}{4}(1+\beta_1\sin^2(\beta_2 t))q^4(t) \quad (8.69)$$

with $\beta_1 > -1$ for the energy to be positive and the voltage expression:

$$v_c(t) = \frac{\delta E_c[q, \alpha]}{\delta q}$$

$$= (1+\beta_1\sin^2(\beta_2 t))q^3(t) \quad (8.70)$$

The dynamics of the network is given by:

$$\dot{q} + Gv_c = \dot{q} + G(1+\beta_1\sin^2(\beta_2 t))q^3(t) = 0 \quad (8.71)$$

Again, we can use the energy left in the capacitor (equation 8.69) as the Lyapunov function. The derivative of the LF along the system trajectory is:

$$\frac{d(LF)}{dt}_{sys. trajec.} = \frac{dE_c}{dt}_{sys. trajec.}$$

$$= \frac{1}{4} \left\{ 2q^3(t)\beta_2\sin(\beta_2 t)\cos(\beta_2 t) + 4(1+\sin^2(\beta_2 t))q^3(t)\dot{q}(t) \right\}$$

$$= -q^4(t) \left\{ G(1+\sin^2(\beta_2 t))^2q^3(t) - \frac{1}{4}\beta_1\beta_2\sin(2\beta_2 t) \right\}$$

$$= -q^4(t) \left\{ G\alpha_1(q,t) - \alpha_2(t) \right\} \quad (8.72a)$$

where

$$\alpha_1(q,t) = (1+\beta_1\sin^2(\beta_2 t))^2q^2(t) \quad (8.72b)$$

and

$$\alpha_2(t) = \frac{1}{4}\beta_1\beta_2\sin(2\beta_2 t) \quad (8.72c)$$

Thus, when the magnitude of $q(t)$ is large, the derivative will be negative. However, when the magnitude of $q(t)$ becomes smaller than some value that is dependent on $\beta_1$, $\beta_2$, and G, the derivative will start toggling from positive to negative and vice versa. Therefore, we cannot say anything conclusively about the absolute stability of the equilibrium point using this LF which happens to be the true energy function. However, being a first order dynamics, we can equate this dynamics to that of a network consisting of a linear unit valued capacitor and a nonlinear time varying capacitor as shown in Fig. 8.18b. Since the resistance in the new circuit is always positive when $G > 0$, we can conclude that the dynamics will be stable as long as $G > 0$. This equivalence also points to another important aspect of this dynamics: since the current through (and hence the power absorption) the nonlinear time varying resistor becomes very small as $q < 1$, the dynamics will not be exponentially stable. We may ask how we can avoid such problems. It is obvious that the problem arises due to the use of a higher power (only) of $q(t)$ for the voltage across the capacitor. Instead, we can include a square term for the energy (resulting in a linear term for the voltage expression) as:

$$E_c[q(t), \alpha(t)] = \alpha(t)q^4(t)$$

$$= \frac{1}{4}(1+\sin^2(t))(kq^2(t) + q^4(t)) \quad ; \quad k > 0 \quad (8.73a)$$

and

Figure 8-18. a) A first-order network with a nonlinear, time-varying capacitor and a LTI resistor; b) Another network with a LTI capacitor and a nonlinear, time-varying resistor with the same dynamics.
\[ v_c(t) = \frac{\delta E_c[q, \alpha]}{\delta q} = \begin{cases} \frac{1}{2} [1 + \sin^2(t)] \{kq(t) + 2q^3(t)\} & (8.73b) \end{cases} \]

The new dynamics is:

\[ \dot{q}(t) + G v_c(t) = \dot{q}(t) + \frac{1}{2} G\{1 + \sin^2(t)\} \{kq(t) + 2q^3(t)\} = 0 \tag{8.74} \]

The derivative of the energy along the system trajectory is:

\[
\frac{d(E_c)}{dt}_{\text{sys. trajec.}} = \frac{dE_c}{dt}_{\text{sys. trajec.}} = \frac{1}{4} \left\{(kq^2(t) + q^4(t))\sin(2t) + (1 + \sin^2(t))(2kq(t) + 4q^3(t))q(t)\right\}_{\text{sys. trajec.}}
\]

\[ = \frac{1}{4} \left\{(kq^2(t) + q^4(t))\sin(2t) - G[(1 + \sin^2(t))(kq(t) + 2q^3(t))]\right\} \]

\[ = -\frac{1}{4} q^2(k + q^2(t)) \left\{ G \left(1 + \sin^2(t)\right) \right\} \frac{\left(1 + \sin^2(t)\right) \left(2q^3(t)\right) \sin(2t)}{G^2 \left(2kq(t) + 4q^3(t)\right) - \sin(2t)} \]

from which we can note that a finite value of \( G \) (for any given positive value of \( k \)) can be chosen so as to ensure that the derivative is negative. Thus, any amount of energy injected into the dynamics by the time-varying mechanism will be absorbed by the resistor forcing \( q(t) \) to reach the equilibrium point. The dynamics is therefore absolutely stable. Also, the presence of the linear term in the voltage expression makes the dynamics exponentially stable as we discussed before.

**Example 10** In this example, we will look at the various equations using a second order network with two linear time varying capacitors as shown in figure 8.19. The two capacitors are described by the two expressions for stored energy as:

\[ E_{c1}[q_1(t), \alpha(t)] = \frac{1}{2} [1 + \alpha_1 e^{-\alpha_1 t}] q_1^2; \quad \alpha_1, \alpha_2 > 0 \]

\[ E_{c2}[q_2(t), \alpha(t)] = \frac{1}{2} (1 + \beta_1 \sin^2(\beta_2 t)) q_2^2(t); \quad \beta_1, \beta_2 > 0 \tag{8.76} \]

with the LF given by the sum of the two stored energy functions. The network dynamics is given by:

\[ \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} -G_1 & -y_{12}[q] \\ -y_{12}[q] & -G_2 \end{bmatrix} \begin{bmatrix} v_{c1}(q_1(t)) \\ v_{c2}(q_2(t)) \end{bmatrix} \tag{8.77} \]

where

\[ v_{c1}[q_1] = \{1 + \alpha_1 e^{-\alpha_1 t}\} q_1 \]

\[ v_{c2}[q_2] = \{1 + \beta_1 \sin^2(\beta_2 t)\} q_2 \tag{8.78} \]

and \( y_{12}[q] \) is some nonlinear function of the state vector \( q \). The derivative of the LF along the system trajectory is given by:

\[
\frac{d(LF)}{dt}_{\text{sys. trajec.}} = \frac{d(E_{c1} + E_{c2})}{dt}_{\text{sys. trajec.}} = -q_1^2 \left\{ G_1 \left(1 + \alpha_1 e^{-\alpha_1 t}\right)^2 + 0.5 \alpha_1 \alpha_2 e^{-\alpha_2 t}\right\} - q_2^2 \left\{ G_2 \left(1 + \beta_1 \sin^2(\beta_2 t)\right)^2 - 0.5 \beta_1 \beta_2 \sin(2\beta_2 t)\right\} \tag{8.79} \]

which again is the difference between the power injected into the network by the time varying mechanism and the power consumed by the two resistors. We find that we can select proper values for the conductances \( G_1 \) and \( G_2 \) that will ensure that the dynamics is stable. We should also note that it is not really...
possible to come up with an equivalent network where the capacitors are linear and the resistors are time-varying.

**Example 11** In the above example, we concluded that proper values for the conductances can be selected such that net power is taken out of the circuit. This is not always the case as we will see in this example. We consider the same circuit of last example with \( G_1 = 0, G_2 = G \), \( y_{12}(q) = -1 \), and the capacitor energy expressions as:

\[
E_{c1}[q_1(t), \alpha(t)] = \frac{1}{2} (1 + \sin^2[t]) q_1^2 \\
E_{c2}[q_2(t), \alpha(t)] = \frac{1}{4} (1 + \cos^2[t]) q_2^4(t)
\] (8.80)

That is, the second capacitor is nonlinear and time-varying. The network dynamics is given by:

\[
\begin{bmatrix}
\dot{q}_1(t) \\
\dot{q}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & -G
\end{bmatrix}
\begin{bmatrix}
v_{c1}[q_1(t)] \\
v_{c2}[q_2(t)]
\end{bmatrix}
\] (8.81)

where

\[
\begin{align*}
v_{c1}[q_1] &= (1 + \sin^2[t]) q_1 \\
v_{c2}[q_2] &= (1 + \cos^2[t]) q_2
\end{align*}
\] (8.82)

are the capacitor voltages. The LF for the dynamics is the sum of the capacitor energy functions. Its derivative along the system trajectory is:

\[
\frac{d(LF)}{dt}_{\text{sys. trajec.}} = \frac{d(E_{c1} + E_{c2})}{dt}_{\text{sys. trajec.}}
\]

\[
= -G(1 + \sin^2[t])^2 q_1^2 + 0.5 q_1^2 - 0.5 q_2^4 \sin[2t]
\] (8.83)

It is not clear from the above expression if a finite value for \( G \) can be found such that the derivative is always negative. Note that here we are using the true energy function and not an arbitrarily chosen LF candidate and still the derivative is not negative-definite.

Upon some reflection, we can conclude that there is no reason to be surprised. After all, we have a time-varying mechanism here and at times it can inject more power than is being consumed making the energy left increase (or equivalently, the derivative becoming positive). The key is that any peak value (or local maxima) that the energy takes at any time is less than the previous peak as shown in Fig. 8.20. Thus, the derivative of the LF has to be negative-definite is only a sufficient condition and not a necessary one. In our case, we have two energy storage devices and the effect of the time-varying phenomena on the energy stored is limited. That is:

\[
E_{c1,\text{Max}}[q_1(t)] = \frac{1}{2} q_1^2 \\
E_{c2,\text{Min}}[q_2(t)] = \frac{1}{4} q_2^4(t)
\] (8.84)

which implies that the capacitors change from one LTI (or NLTI in the case of the second capacitor) element to another. For a given value of \( q_1(t) \) and \( q_2(t) \), once the time-varying mechanism pushes the energy envelope to the maximum level, there is nothing much it can do. It can only take back some of the energy it gave to the elements. Therefore, for all practical purposes, the circuit behaves like a nonlinear circuit. The linear damper (as opposed to a nonlinear or time-varying damper) will consume power uniformly making the dynamics asymptotically stable.

This example, demonstrates how we can make complex time-varying dynamics and at the same time preserve stability. We can make the damper time-varying as well leading to exotic behavior. It is fair to say that we have just scratched the surface and much work needs to be done.
Example 12-14 We now consider the three second-order dynamics used in chapter 4 to illustrate different limit cycle behaviors and explain their behavior using their network equivalence. The dynamics are:

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
- \begin{bmatrix}
x_1(t)\{x_1^2(t) + x_2^2(t) - 1\} \\
x_2(t)\{x_1^2(t) + x_2^2(t) - 1\}
\end{bmatrix}
\tag{8.85a}
\]

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
- \begin{bmatrix}
x_1(t)\{1 - x_1^2(t) - x_2^2(t)\} \\
x_2(t)\{1 - x_1^2(t) - x_2^2(t)\}
\end{bmatrix}
\tag{8.85b}
\]

and

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
- \begin{bmatrix}
x_1(t)\{|x_1(t)| + x_2(t)\} \\
x_2(t)\{|x_1(t)| - x_2(t)\}
\end{bmatrix}
\tag{8.85c}
\]

It is easy to see that a linear time varying network architecture of the form shown in Fig. 8.21 with two unit-valued LTI capacitors, a two-port LTI lossless gyrator with an admittance matrix as shown in the figure and two time varying resistors with the same conductance values at both ports:

\[
G(t) = G_1(t) = x_1^2(t) + x_2^2(t) - 1
\tag{8.86a}
\]

\[
G(t) = G_2(t) = 1 - x_1^2(t) - x_2^2(t)
\tag{8.86b}
\]

and

\[
G(t) = G_3(t) = |1 - x_1^2(t) - x_2^2(t)|
\tag{8.86c}
\]

leads respectively to the three dynamics given in (8.80). The three conductances become zero when

\[
x_1^2(t) + x_2^2(t) = 1
\tag{8.87}
\]

leading to a lossless LC network. The parameters of this lossless network are such that the resulting oscillation is given by:

\[
x_1^2(t) + x_2^2(t) = r^2
\tag{8.88}
\]

where \(r\) is a constant that will depend on the initial charge on the capacitors. This expression includes the expression (8.87) as a special case. Hence, the limit cycle oscillation.

In the network corresponding to the first dynamics, the conductances become passive when \(x_1^2(t) + x_2^2(t) > 1\) forcing \(E(t) = 0.5(x_1^2(t) + x_2^2(t))\), the remaining energy in the circuit, to decrease. When \(x_1^2(t) + x_2^2(t) < 1\), the conductances become negative forcing \(E(t) = 0.5(x_1^2(t) + x_2^2(t))\), the remaining energy in the circuit, to increase. Therefore, the circle of radius one in the phase plane becomes the attracting limit cycle for the first dynamics. It should be noted that the behavior of the time-varying resistors, changing from a passive one to an active one and back, has more influence on the circuit dynamics than the initial energy stored in the capacitors and therefore a limit cycle oscillation that is independent of the initial conditions.

In the network corresponding to the second dynamics, the conductances become negative when \(x_1^2(t) + x_2^2(t) > 1\) forcing \(E(t) = 0.5(x_1^2(t) + x_2^2(t))\), the remaining energy in the circuit, to increase. When \(x_1^2(t) + x_2^2(t) < 1\), the conductances become positive forcing \(E(t) = 0.5(x_1^2(t) + x_2^2(t))\), the remaining energy in the circuit, to decrease. Therefore, the limit cycle becomes an unstable one. The origin and infinity are the two stable equilibrium points for this dynamics.

In the network corresponding to the third dynamics, the conductances become passive when \(x_1^2(t) + x_2^2(t) > 1\) as well as when \(x_1^2(t) + x_2^2(t) < 1\) forcing \(E(t) = 0.5(x_1^2(t) + x_2^2(t))\), the remaining energy in the circuit, to increase. Therefore, the circle of radius in the phase plane becomes a semi-stable limit cycle.
As we noted, in all the three dynamics in (8.85), the state trajectory for which the time-varying resistance becomes a open circuit also happens to be the one solution for the remaining lossless LTI circuit. We may ask if this is really necessary for limit cycle oscillation to exist. The answer is no as we show using a new example. The chosen dynamics is:

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
-4 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
- \begin{bmatrix}
x_1(t)\{x_1^2(t) + x_2^2(t) - 1\} \\
x_2(t)\{x_1^2(t) + x_2^2(t) - 1\}
\end{bmatrix}
\tag{8.89a}
\]

which differs from expression (8.85a) corresponding to stable limit cycle dynamics by just one parameter in the constant matrix (2,1 element). The dynamics can be re-written as:

\[
\begin{bmatrix}
\dot{x}_1(t) \\
0.25 \dot{x}_2(t)
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
- \begin{bmatrix}
x_1(t)\{x_1^2(t) + x_2^2(t) - 1\} \\
0.25 x_2(t)\{x_1^2(t) + x_2^2(t) - 1\}
\end{bmatrix}
\tag{8.89b}
\]

which again points to a network with two capacitors (one of unit value and the other 0.25 F), a lossless two-port LTI gyrator and two time-varying conductances. The two conductances still become zero (open circuit) on the unit circle in the phase plane (trajectory # 1 in figure 8.22). On the other hand, the solution of the rest of the LTI lossless circuit is given by:

\[x_1^2(t) + 0.25 x_2^2(t) = r^2\tag{8.90}\]

which doesn’t include the unit circle. We show two of these trajectories (# 2 and # 3) when \(r = 1\) and 0.5 in the figure. We also show the response of the dynamics as trajectory # 4. The limit cycle oscillation is neither the unit cycle nor any of the ellipsoids described by (8.90). Instead, we get a response that goes inside of the unit circle for some time (A to B and C to D in the figure) when the conductances are negative and outside of the unit circle for rest of the cycle (B to C and D to A in the figure) when the conductances are positive. That is, the time-varying resistors’ characteristics dominate the dynamics. As there are no other passive resistors in the circuit, the capacitors get charged for part of the cycle and discharge for the rest of the cycle. The various parameters and the terms in the nonlinear dynamics determine the exact shape of the response.

The above example demonstrates how we can use network concepts to explain easily the behavior of complex dynamics and or to obtain dynamics with the required characteristics. In a later chapter, we will use similar techniques to explain further the concepts of chaos and fractals.

Figure 8-22. The phase-plane response of the nonlinear dynamics of equation (8.84) which exhibits limit cycle behavior (State variable \(x_2(t)\) is along the x axis and state variable \(x_1(t)\) is along the y axis). The response is influenced by the trajectory # 1 which separates the regions in which the time-varying conductances become passive and active, and not on the response of the LTI lossless network that results when the conductances become zero.

8.5 Summary

In this chapter we learnt that the phenomena of time-variance can be introduced properly at the element level using the concepts of power and energy. We first studied time-varying resistors that can be linear or nonlinear, fully passive or fully active (negative) or non-passive. We saw the definitions for time-varying gyrators and transformers, multi-port devices that are also lossless. We then looked at time-varying capacitors and inductors, dynamic devices that are also lossless. Proper interconnection of these building blocks lead to complex time-varying circuits. Using KCL and KVL one can write the dynamics corresponding to such circuits. From the network elements, we can write the true energy function as the Lyapunov function and the derivative of the LF along the
system trajectory easily by inspection. A number of first- and second-order circuits were provided to illustrate the power of the new approach. This approach allows us to correctly differentiate nonlinear dynamics from time-varying dynamics. From the examples provided, it became clear that the LF need not be a time-varying function even if the dynamics is time-varying. Also, we can have time-varying dynamics in which the true LF is not monotonically decreasing (the derivative along the system trajectory not negative-definite or negative-semi-definite) but still the dynamics is absolutely stable. Though the examples provided are limited to second-order dynamics, it should be noted that the approach is applicable to any higher-order systems. Further, the building block approach can be of tremendous help in a number of applications as we will see in part II of this book.