6. CIRCUITS MADE OF LINEAR AND NONLINEAR (TIME-ININVARIANT) ELECTRICAL ELEMENTS & THEIR DYNAMICS

6.1 Introduction to the chapter

In this chapter, we consider the use of the various electrical-circuit-building-blocks introduced in the last chapter in forming complex electrical networks and study the resulting dynamics. We first consider circuits made of LTI elements, the basic laws (Kirchhoff’s current and voltage law and Tellegen’s theorem) that apply when such elements are interconnected, and the restrictions that need to be observed while interconnecting such elements. We consider the I/O characteristics of such networks and discuss concepts such as impedance and admittance functions and positive real functions. We will find that networks with lossy elements lead to stable circuits. Since the lossy elements are also linear (which imply power consumption ability that is proportional to the square of the current or voltage), we will find that such networks also have the bounded-input, bounded-output property. We will consider building complex multi-port networks in a systematic manner and study their impedance and admittance matrices. We will find that a state-space representation of such a network carry more information (the I/O characteristics of the individual elements & the structural information) than is indicated by stability alone. We also discuss two important techniques in LTI network theory, that of impedance scaling and frequency transformations and their importance in LTI signal processing. We also make a connection between results from linear circuit theory and mathematical systems theory and point out the differences from the network perspective.

6.2 Circuits made of linear, time-invariant Passive Elements

In this section, we consider circuits made of linear passive elements, rules to be observed in making such circuits, properties of electrical circuits made of passive elements and the sufficiency of existing elements in providing certain kinds of responses, and various theorems that are useful in characterizing and/or analyzing electrical circuits. We further discuss issues such as impedance scaling and frequency transformations and their importance in LTI signal processing. We also make a connection between results from linear circuit theory and mathematical systems theory and point out the differences from the network perspective.

6.2.1 Kirchhoff’s current and voltage law and Tellegen’s theorem

An electrical circuit or network is a collection of electrical elements connected or coupled together. A linear passive network is formed by connecting the linear passive elements, resistors, capacitors, inductors and the transformers. An example of a linear passive network connected to a voltage source is shown in Fig. 6.1. The network consists of six one-port elements and one two-port element, all assumed to be linear and time-invariant. From element v-i relationship perspective, a two-port element leads to two equations (an M-port element leads to M equations). When we define the number of elements, denoted as ‘b’, we will talk in terms of equivalent one-port elements or equivalently the number of equations that arise from the elements. Thus, for our example, b will be eight (six one-port elements plus two times the number of two-port elements).
The connection should lead to a number of nodes (or points at which one terminal of a number of elements are connected together; numbered $n_1$ to $n_8$ in fig. 6.1) such that any node can be reached from any other node in the network by traversing a path through the network elements. For such a connected network, we can choose any one of these nodes as the reference or datum node for measuring electric potentials and define the voltage at other nodes with reference to this datum node. Similarly, we can assign arbitrarily the signs for branch voltages (that is, the voltage across the network elements) and a reference direction for branch currents as shown in the figure.

Given any connected passive network with the voltages and the currents defined as indicated above, two fundamentals laws help to quantify the interconnection properties of the network. The first one is the Kirchoff’s current law (KCL) based on the fundamental law of physics on the electric charge conservation (no net charge is either created or destroyed). KCL states that the algebraic sum of currents entering (or leaving) any node of a lumped circuit at any (and all) time $t$ is exactly equal to zero.

The application of the KCL to nodes $n_1$ to $n_8$ leads to the following six equations written in the matrix form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
i_1(t) \\
i_2(t) \\
i_3(t) \\
i_4(t) \\
i_5(t) \\
i_6(t) \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
i_7(t) \\
i_8(t) \\
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
v_1(t) \\
v_2(t) \\
v_3(t) \\
v_4(t) \\
v_5(t) \\
v_6(t) \\
v_7(t) \\
v_8(t) \\
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
i_1(t) \\
i_2(t) \\
i_3(t) \\
i_4(t) \\
i_5(t) \\
i_6(t) \\
i_7(t) \\
i_8(t) \\
\end{bmatrix}
\]

where $i=[i_1(t), i_2(t), \ldots, i_8(t)]^T$ is a vector branch of currents and $A$ is known as the incidence matrix whose entries are restricted to $+1, -1$ or 0. Thus KCL reflects the interconnection properties of the network and not the properties of the elements forming the network. Also it should be noted that the last two equations in the KCL are linearly dependent on the other four equations and hence do not convey any new information. In general for any connected network with $n$ nodes and $n_T$ number of Transformers and $n_G$ number of two-port gyrators, the KCL equations from $n-1-n_T-n_G$ nodes (obtained by using only 2 nodes out of the 4 nodes formed at the interconnection of the two-port elements) will form a set of linearly independent equations that will characterize completely the interconnection property of the network.

Kirchoff’s voltage law (KVL) is the second fundamental law that describes the interconnection properties and uses loops or the sequence of closed nodes in the network. In the figure 6.1, we have seven loops, $l_1$ to $l_7$. KVL states that for all connected circuits and for all times $t$, the algebraic sum of all node-to-node voltages around any closed node sequence is equal to zero. For the network in fig. 6.1 with the voltage across the various elements as defined and for the seven closed loops shown, we can write seven KVL equations as:

\[
\begin{bmatrix}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
v_1(t) \\
v_2(t) \\
v_3(t) \\
v_4(t) \\
v_5(t) \\
v_6(t) \\
v_7(t) \\
\end{bmatrix}
= 0
\]
We can note that the last three equations are linearly dependent on the first four equations. The number of independent equations obtained from KCL or KVL is the same, equal to four here. Again, KVL holds regardless of the properties of the elements forming the network.

**Tellegen’s theorem** (TT) is another theorem that is very general and useful and follows directly from the Kirchoff’s laws. Let us explain the Tellegen's theorem and its importance again by using the network example in fig. 6.1. With the branch currents and voltages as defined in the figure and satisfying KVL, \( t_1 \) and \( t_2 \) denoting two instances of time (\( t_1 \) and \( t_2 \) may or may not be the same), according to Tellegen’s theorem:

\[
\sum_{k=1}^{b} v_k(t_1) i_k(t_2) = 0
\]  

That is, the current in the L-th element is constrained to a value equal to the algebraic sum of the currents in the other elements, regardless of the elements’ properties. On the other hand, we have noted that the current in a linear inductor has to be continuous and hence has to remain independent. Combining these two properties we find that a possibility for conflict arises when we have a node with only inductors and or independent current sources. Thus from a design perspective, nodes with only inductors and or independent current sources have to be avoided. Similarly, from a consideration of KVL applied to a loop as shown in Fig. 6.2b and properties of linear capacitors, we find that formation of loops with only capacitors and or independent voltage sources have to be avoided. We will assume that these two rules will always be obeyed.

6.2.2 Restrictions on the Interconnections:

Having defined the KCL and KVL (and TT), and knowing the properties of the individual circuit elements, we are in a position to look for any caution that needs to be exercised while interconnecting the elements to form a circuit. Consider fig. 6.2a where we have shown one node \( n_j \) (of perhaps a complex circuit) where each terminal of \( e_{nj} \) elements are connected. We can apply the KCL to this node to arrive at:

\[
\sum_{k=1}^{e_{nj}} i_k(t) = 0 \quad \text{for} \quad j = 1, 2, \cdots, n_a
\]  

which holds regardless of the types of elements that are connected to this node. The above equation can be rewritten as:

\[
i_{jL} = - \sum_{k=1, k \neq L}^{e_{nj}} i_k(t) \quad \text{for} \quad j = 1, 2, \cdots, n_a
\]

That is, the current in the L-th element is constrained to a value equal to the algebraic sum of the currents in the other elements, regardless of the elements’ properties. On the other hand, we have noted that the current in a linear inductor has to be continuous and hence has to remain independent. Combining these two properties we find that a possibility for conflict arises when we have a node with only inductors and or independent current sources. Thus from a design perspective, nodes with only inductors and or independent current sources have to be avoided. Similarly, from a consideration of KVL applied to a loop as shown in Fig. 6.2b and properties of linear capacitors, we find that formation of loops with only capacitors and or independent voltage sources have to be avoided. We will assume that these two rules will always be obeyed.

6.2.3 I/O Characteristics of linear passive Networks:

The interconnections of linear passive elements leads to networks (continuous systems) with the linear and time invariant property. Thus, we can characterize the I/O behavior of a linear passive circuit using concepts such as impulse response, transfer function (ratio of the LT of the output to the LT of the input) and the frequency response.
Application of the Kirchhoff’s laws and the v-i relationship for the various elements either in the time domain or the frequency domain will lead respectively to a set of linear differential equations or linear equations in the complex frequency variable ‘s’. They can be solved further to obtain the constant coefficient linear differential equation connecting a particular output with the input or the transfer function in the frequency domain. For example, for the example network in fig. 6.1, using the KVL for loops 1 and 2, KCL for nodes 2 and 3, and all the elements v-i properties, we obtain, as time-domain equations:

\[ v_1(t) = L_1 \frac{di_1(t)}{dt} = v_s - v_2 \]

\[ i_2(t) = c_2 \frac{dv_2(t)}{dt} = i_1 - i_3 \]

\[ v_3(t) = L_3 \frac{di_3(t)}{dt} = v_2 - v_4 \]

\[ i_4(t) = c_4 \frac{dv_4(t)}{dt} = i_3 - i_5 \]

\[ v_5(t) = v_4(t) \]

\[ v_7(t) = v_6(t) \]

\[ v_6(t) = N v_5(t) \]

\[ v_7(t) = R_1 i_7(t) \]

\[ i_7(t) = -i_6(t) = \frac{1}{N} i_5(t) \]

or, in the frequency domain (assuming no initial energy in the reactive elements):

\[ V_1(s) = L_1 I_1(s) = V_s(s) - V_3(s) \]

\[ I_2(s) = c_2 V_2(s) = I_1(s) - I_3(s) \]

\[ V_3(s) = L_3 I_3(s) = V_2(s) - V_4(s) \]

\[ I_4(s) = c_4 V_4(s) = I_3(s) - I_5(s) \]

\[ V_5(s) = V_4(s) \]

\[ V_6(s) = N V_5(s) \]

\[ V_7(s) = R_1 I_7(s) \]

\[ I_7(s) = -I_6(s) = \frac{1}{N} I_5(s) \]

That is, we have four dynamic equations (same as the number of reactive elements in the circuit) and four algebraic equations. The equations could be solved to obtain four dynamic equations in the four variables \( i_1(t), v_2(t), i_4(t) \) and \( v_4(t) \), the currents through the inductive elements and the voltages across the capacitive elements and can be written in the state space form as:

\[
\begin{bmatrix}
L_1 \frac{di_1(t)}{dt} \\
\frac{dv_2(t)}{dt} \\
L_3 \frac{di_3(t)}{dt} \\
\frac{dv_4(t)}{dt}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/N^2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
i_1(t) \\
v_2(t) \\
i_3(t) \\
v_4(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\] (6.7a)

or

\[
\begin{bmatrix}
\frac{di_1(t)}{dt} \\
\frac{dv_2(t)}{dt} \\
\frac{di_3(t)}{dt} \\
\frac{dv_4(t)}{dt}
\end{bmatrix} =
\begin{bmatrix}
L_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/N^2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
i_1(t) \\
v_2(t) \\
i_3(t) \\
v_4(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\] (6.7b)

or, using vector-matrix notation:

\[
\dot{x}(t) = Ax(t) + u(t)
\] (6.8a)

or in a generalized form:

\[
\dot{x}(t) = Ax(t) + Bu(t)
\] (6.8b)

where \( x(t) = [i_1(t), v_2(t), i_4(t), v_4(t)]^T \) is the state vector of size 4 (n in general), and the matrix A is of dimension 4 by 4 (n x n). A number of important observations can be made from this example. First, the network is shown with only one input source (a voltage source) and can be converted to a multi-input network by inserting the inputs properly. This will change only the entries in the vector u containing the source(s) in the state space representation and the entries in the matrix B that multiplies the vector of input sources. This is one very useful property of the state space representation where the representation remains practically the same whether we deal with single input, single output system or multi-input, multi-output system. Secondly, we can note that there are four
reactive elements (2 inductors and 2 capacitors) and there are four equations in the state space representation. If we compute the determinant of the matrix $A$, we will find that it will be non zero as long as the element values are non zero. That is, the equations are linearly independent. Thus, if we solve the state-space equations for the I/O relationship we will obtain a fourth order constant coefficient differential equation or a fourth order transfer function as we will show later.

The property that the order of the system equals the number of reactive elements will good as long as the there are no nodes with only inductors and current sources or loops with only capacitors and voltage sources.

We have to be careful while using the generalized state space model given in (6.8). Otherwise we will end up with models that do not correspond to specific physical systems. For example, if we look at the $A$ matrix corresponding to the passive network of Fig. 6.1, we will note that -$A$ is positive semi-definite. This is one important property of passive networks. Also, there are restrictions on the $b$ vector (or the $B$ matrix). Considering the single input case, we can rewrite the general model as:

$$\dot{x}(t) = Ax(t) + bv_s(t)$$  \hspace{1cm} (6.9)$$

where $b = \begin{bmatrix} \mathbf{y}L_1 & 0 & 0 & 0 \end{bmatrix}$ for our network of Fig. 6.1. If the same source $v_s(t)$ is to be connected to the other state equations, from a mathematical control theory perspective, the zero valued entries in the vector $b$ will be simply changed. Such an approach fails to show the interactions or loading that will result. For example, consider changing the $b$ vector to $\begin{bmatrix} \mathbf{y}L_1 & -1 & 0 & -1 \end{bmatrix}^T$ to indicate that the same source affects the second and the fourth state equations as well. The corresponding network architecture will be as shown in Fig. 6.3a, where we have assumed the existence of two controlled current sources $i_u(t)$ and $i_{u2}(t)$ with the following relationships:

$$i_u(t) = \frac{1}{c_2} v_s(t)$$

$$i_{u2}(t) = \frac{1}{c_2} v_s(t)$$  \hspace{1cm} (6.10)$$

That is, the model now corresponds to an active circuit.

If the same source has to affect the second and fourth state equations and at the same time the network has to remain passive, the network in Fig. 6.1 may be changed to the one in Fig. 6.3b where we have added two more resistors to produce the interaction. It can be shown that the $b$ vector associated with the state equations is $\begin{bmatrix} \mathbf{y}L_1 & \mathbf{y}R_1 & 0 & \mathbf{y}R_2 \end{bmatrix}^T$. That is, the signs of the elements have to be positive and we cannot make the third element non-zero. In subsection 6.1.4, we will look at the general form of LTI state space dynamics obtained from passive networks. We will find that the properties of passivity and proper interconnectivity will be reflected in the form the matrices $A$ and $B$ take.

If we consider the two I/O transformations $\{v_s(t) \rightarrow i_u(t)\}$, and $\{v_s(t) \rightarrow v_{s2}(t)\}$ we find that the first represents a voltage to current and the second a voltage to voltage transformation. Further, the I/O transformation $v_s(t) \rightarrow i_u(t)$ involves voltage and current associated with the same two terminals (same port) where as the I/O transformation $v_s(t) \rightarrow v_{s2}(t)$ involves voltages at different ports. The first transformation goes with the terminology "driving-point characteristics" where as the latter is known as "transfer characteristics". Because of this difference, there are differences in the forms each characteristic can take as we will see now.

We can solve for the constant coefficient linear differential equations for the two transformations and obtain:

$$\begin{bmatrix} R_L c_4 L_2 c_2 L_4 \frac{d^4}{dt^4} + N^2 L_3 c_2 L_4 \frac{d^3}{dt^3} + R_L (c_4 L_2 + c_4 L_1 + c_4 L_3) \frac{d^2}{dt^2} + N^2 (L_2 + L_3) \frac{d}{dt} + R_L \end{bmatrix} i_u(t)$$

$$= \begin{bmatrix} R_L c_4 L_3 c_2 \frac{d^3}{dt^3} + N^2 L_3 c_2 \frac{d^2}{dt^2} + R_L (c_2 + c_3) \frac{d}{dt} + N^2 \end{bmatrix} v_s(t)$$  \hspace{1cm} (6.11a)$$

\[1\] We can obtain by including a current source two other possible I/O transformations: current to current and voltage to current transformations.
Equivalently in the frequency domain,

\[
\begin{align*}
I(s) &= Y(s) \\
\frac{V(s)}{V_i(s)} &= \frac{R_L c_4 L_3 c_3 s^3 + N^2 L_3 c_3 s^2 + R_L (c_2 + c_4) s + N^2}{R_L c_4 L_3 c_3 L_1 s^4 + N^2 L_3 c_3 L_1 s^3 + R_L (c_1 L_1 + c_4 L_1 + c_4 L_3) s^2} \\
&\quad + N^2 (L_3 + L_1) s + R_L
\end{align*}
\]  

(6.12a)

\[
\frac{V(s)}{V_i(s)} = \frac{R_L N}{R_L c_4 L_3 c_3 L_1 s^4 + N^2 L_3 c_3 L_1 s^3 + R_L (c_1 L_1 + c_4 L_1 + c_4 L_3) s^2} \\
&\quad + N^2 (L_3 + L_1) s + R_L
\]  

(6.12b)

where the driving point function \(Y(s)\) is also known as the admittance function and its inverse \(Z(s)\) another driving point function is known as the impedance function. The driving point functions obtained from a network containing only lossless elements are known as the reactance or the susceptance functions. It can be noted that the denominators of the \(Y(s)\) (a driving point function) and \(T(s)\) (a transfer function) are the same (equal to the determinant of the matrix, \(sI - A\) where \(I\) is the identity matrix) and will remain so even when there are multiple inputs.

We noted that networks made of linear passive elements are examples of continuous systems with the LTI property. Thus, we may ask if all I/O characteristics of a stable LTI systems describable by constant coefficient linear differential equation can be realized as either the “driving point characteristics” and/or the “transfer characteristics” of such a network. We may recall from chapter 2 that the transfer functions of such LTI systems are rational in the complex variable \(s\) with the poles (zeros of the denominator polynomial) restricted to the left side of the \(s\)-plane. The zeros of the numerator polynomial can be anywhere on the \(s\)-plane and complex poles and the zero appearing in the complex conjugate form so that the polynomials have only real coefficients. That is, the denominator polynomial has to be strictly Hurwitz where as the numerator polynomial can be anything.

It is well known in network theory that all I/O characteristics of stable LTI systems (with in a scaling factor) can be realized as the transfer characteristics (and not driving point characteristics) of a passive networks made up of the four elements: \(R\), \(L\), \(C\), and Brune transformer (BT). That is, these four elements form a complete set. If either the inductors or the capacitors are missing, the
realizable characteristics are restricted to ones with transfer functions having only real poles and zeros. Since we know that the I/O characteristics of an inductor can be simulated by a gyrator, capacitor combination, we can say that the four elements R, C, Gyrator and the Brune transformer form another complete set. A number of networks, the corresponding transfer functions and their important properties are listed in Table 6.1.

In the table, one may notice architectures such as the two port L, C networks (lossless networks) connected to the source at one port and resistive load at the other point, and ladder and lattice architectures. These architectures are well known for certain desirable properties such as a) the transfer characteristics being very insensitive to the elements' values variations\(^2\), b) the ease with which transmission zeros can be placed at desirable frequencies and c) the use of lattice architectures in producing right hand side zeros for the transfer function. We have also shown a *doubly terminated network* where in addition to the load resistor, there is also a resistor in series with the ideal voltage source. This resistor is included since in a real world there are no ideal sources as we have defined before.

Driving point characteristics and driving point functions form a much restricted set of LTI systems response functions within the class of I/O functions represented as constant coefficient linear differential equations. This is to be expected as we place restrictions as to where the response has to be measured. Let us summarize the salient results through the use of the simple one-port networks shown in Table 6.1. In the table, we have shown networks consisting of only L, C elements (lossless net), R, L, C elements (lossy net), and R, L, C, and Brune transformer elements. We have also indicated the driving point functions and the resulting pole, zero locations. Many observations can be made from these networks (which have been proven to be valid in network synthesis literature). First, it can be noted from nets 1 - 3 that for networks made up of L, C and L, C, BT elements:

1) The driving point functions are ratios of odd degree polynomials to even degree polynomials with real coefficients (or even degree to odd);
2) The absolute value of the difference in the degree of the numerator and the denominator polynomials is restricted to one;
3) The order of the transfer function is given by the number of L, C and BT elements;
4) The poles and the zeros are confined to the imaginary \((j\omega)\) axis of the s-plane [direct consequence of property # 1].

\(^2\) That is, reasonable variations in the elements' values (that is unavoidable due to factors such as variations in the parameters in the fabrication process) will result in only a minimal variation in the I/O characteristic. In fact, this property has made such architectures popular in the digital filter realization as well.

<table>
<thead>
<tr>
<th>No.</th>
<th>Network &amp; Driving point function (DPF)</th>
<th>Pole-Zero location in the s-plane (0=zero; x=pole)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( i_1(t) \rightarrow L \rightarrow C \rightarrow Y_1(s) = \frac{V_1(s)}{I_1(s)} = \frac{LCs^2 + 1}{Cs} )</td>
<td>( \alpha ) ( \beta ) ( \gamma )</td>
<td>Lossless net; Poles &amp; zeros confined to the imaginary axis of the s-plane, are simple &amp; alternate. DPF ratio of odd to even (or even to odd) polynomials</td>
</tr>
<tr>
<td>2</td>
<td>( + L_1 \rightarrow i_2(t) \rightarrow C_2 \rightarrow Y_2(s) = \frac{I_2(s)}{V_2(s)} = \frac{L_sC_s}{C_2s^2 + (L_s+L_2+C_4)s + 1} )</td>
<td>( \alpha ) ( \beta ) ( \gamma )</td>
<td>Same as above; (A ladder net).</td>
</tr>
<tr>
<td>3</td>
<td>( + L_{11} \rightarrow i_3(t) \rightarrow C_3 \rightarrow Y_3(s) = \frac{I_3(s)}{V_3(s)} = \frac{C_sL_s}{(L_s+L_{11})s^2 + (L_s+C_4)s + 1} )</td>
<td>( \alpha ) ( \beta ) ( \gamma )</td>
<td>Lossless net with a Brune transformer (BT). Each BT increases the degree by one. The DPF obeys the properties of lossless nets (Nets 1 &amp; 2)</td>
</tr>
</tbody>
</table>

Table 6-1. Networks made of LTI elements and their properties.
<table>
<thead>
<tr>
<th>No.</th>
<th>Network examples &amp; DPFs.</th>
<th>Pole-Zero locations</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td><img src="image" alt="Network example" /></td>
<td>$Z_4(s) = \frac{V_4(s)}{I_4(s)} = \frac{Ls+R}{LCs^2 + RCs + 1}$</td>
<td>Lossy RLC network; poles and zeros on the LHS of s-plane. $Z_4(s)$ is BIBO stable; $Y_4(s)$ is not (pole at s=0).</td>
</tr>
<tr>
<td>5</td>
<td><img src="image" alt="Network example" /></td>
<td>$Y_5(s) = \frac{I_5(s)}{V_5(s)} = \frac{LCs^2 + 1}{RLCs^2 + Ls + R}$</td>
<td>Lossy RLC net. Poles of the impedance function on $\Re{s}$ axis &amp; hence is not BIBO stable. Its inverse, the admittance function (adm. fn.) is BIBO stable.</td>
</tr>
<tr>
<td>6</td>
<td><img src="image" alt="Network example" /></td>
<td>$Z_6(s) = \frac{L_2C_2L_3s^3 + L_2C_3s^2 + (L_1 + L_2)s + 1}{LC_2s^2 + C_2s + 1}$</td>
<td>Ladder LC net with resistive termination. Poles &amp; zeros on LHS of s-plane. The admittance function is BIBO stable but the impedance fn. is not due to pole at infinity.</td>
</tr>
<tr>
<td>7</td>
<td><img src="image" alt="Network example" /></td>
<td>$Z_7(s) = \frac{L_2C_2L_3s^3 + L_2C_3s^2 + (L_1 + L_2)s + 1}{LC_2s^2 + C_2s + 1}$</td>
<td>A resistive terminated lossless net with a BT in the input end. Both the admittance and the impedance functions are BIBO stable and represent driving point functions known as minimal functions.</td>
</tr>
</tbody>
</table>

$\Rightarrow$ A minimal fn. ($Z_7(j1) = 0.5j$ & Real $Z_7(j\omega) > 0$ for all other $\omega$)

---

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<td>Another RLC net with a BT in the input end and realizing second-order DPF. Both the DPFs are BIBO stable.</td>
</tr>
</tbody>
</table>

$\Rightarrow$ A minimal fn. ($Z_{11}(j1) = -j$ & Real $Z_{11}(j\omega) > 0$ for all other $\omega$)

---

Table 6-1 (Contd.)

Table 6-1 (Contd.)
When the Brune transformer is present (3-rd net), the equivalent circuit with only L and C elements include a reactive element with negative value as we have seen before when we discussed two port coupled inductors. When a resistance is included in otherwise lossless network (net 4 onwards), the poles (and or the zeros) move to the left hand side of the s-plane (Real[s]<0) depending on the network configuration. That is, the numerator and or the denominator polynomials of such networks are strictly Hurwitz and some of the I/O transformations as defined in the networks are BIBO stable. For example, $Z_4(s)$, $Y_5(s)$, $Y_6(s)$ of networks 4 to 6 are BIBO stable (denominator polynomials strictly Hurwitz and the degree of the numerator polynomials less than that of the corresponding denominator polynomials) where as $Y_4(s)$, $Z_4(s)$, $Z_6(s)$, the corresponding inverses, are not (These driving functions have poles on the imaginary axis or at $s=\infty$). On the other hand, $Y_7(s)$ and $Z_11(s)$ as well as $Y_7(s)$ and $Y_{11}(s)$ (the corresponding inverses) of nets 7 and 11 are BIBO stable. From the network's perspective, the difference between networks 7 and 11 and 4 to 6 is that the first element in 7 and 11 is a Brune transformer where as the networks 4 to 6 have an inductor or a capacitor as the first element.

Networks 7 to 9 have the same driving point functions , $Z(s)$ and $Y(s)$. Network 7 uses a BT whereas network 8 contains a negative valued inductor. Network 9 realizes the same second order driving point function using no Brune transformer, but the price paid is an increase in the number of reactive elements. That is, it is a non minimal realization of the second order DPF in the terms of reactive elements (the order of the transfer function is less than the number of reactive elements in the circuit). Network 10 depicts a general network architecture, a balanced bridge, on which network 9 is based.

Networks 11 and 12 have the same driving point function, $Z(s)$, and differs in the location of the negative inductor. Here again, $Z(s)$ as well as $Y(s)$ are BIBO stable. Finally, network 14 is obtained by adding in series a parallel $L$, $C$ combination to network 7 that makes the driving point functions $Z_{14}(s)$and $Y_{14}(s)$ marginally stable.

We can show that for all the driving point functions obtained from the networks:

1) $Z(s)$ and $Y(s)$ real when s is real
2) $\frac{\text{Real}[Z(s)]}{\text{Real}[Y(s)]} \geq 0$ when $\text{Real}[s] \geq 0$ \hspace{1cm} (6.13)

This definition is not equivalent to

$\text{Real}[Z(s)] > 0$ when $\text{Real}[s] > 0$
$\text{Real}[Y(s)] = 0$ when $\text{Real}[s] = 0$ \hspace{1cm} (6.14)
as interpreted in some control theory literature. For example, for networks 1 to 4 (L, C, BT networks), the above interpretation (equation 6.14) holds where as for network 7 we find:

\[
\text{Real}\left[Z_7(j\omega)\right] = \frac{(\omega^2 - 1)^2}{\omega^4 - 3\omega^2 + 4}
\]

which becomes exactly equal to zero at \( \omega = 1 \) and is positive for all other values of \( \omega \).

A rational function \( P(s) \) that satisfies the conditions given in (6.13) is known as a positive real (PR) function. It is well known in network theory that:

1. Driving point functions (both impedance and admittance) of a passive network made of R, L, C and BT elements are positive real and
2. Any positive real function \( P(s) \) can be realized as the admittance or the impedance function of a passive network consisting of those four elements. The various important properties of a PR functions are summarized in table 6.2.

Positive real functions such as \( Z_7(s) \) have some specific properties. In addition to satisfying the conditions for PR, they are characterized by:

1. No poles or zeros on the imaginary axis of the s plane (both the numerator and the denominator polynomials are strictly Hurwitz);
2. Finite, real, and positive values at \( s = 0 \) and \( s = \infty \). (the degrees of the numerator and the denominator are the same and both have non-zero valued constant term);
3. The real part, \( \text{Real}[Z(j\omega)] \), vanishing for at least one finite frequency, \( \omega_1 \), such that \( Z(j\omega_1) = 0 \pm jx_1 \) with \( x_1 \neq 0 \).

Such a positive real function is known as a minimum function. The inverse of a minimal function is also minimal. Both the minimal function and its inverse are BIBO stable which is not true for all positive real functions. As we noted from the examples, some positive real functions can be BIBO stable while their inverses are not BIBO stable.

Finally, from the various I/O transformations (which are not driving point transformations) as indicated for the various networks in the table 6.1, we can observe that when the network contains at least one resistor making the network lossy, the transfer functions have the following properties:

1. The transfer function poles are restricted to the left hand side of the s-plane (strictly Hurwitz denominator polynomial) and the degree of the denominator polynomial is less than or equal to the degree of the numerator polynomial. Thus, the transfer functions of lossy networks are BIBO stable.
2. The numerator polynomials: a) can simply be constants (leading
Properties of Positive Real Function, P(s)
(with no common factors in the numerator & the denominator)

1. Necessary conditions:
   a) The coefficients of the numerator and the denominator polynomials of P(s) be real and positive.
   b) The degrees of the numerator and the denominator polynomials differ at most by one.
   c) Numerator and denominator terms of lowest degree differ at most by 1.
   d) Imaginary axis poles and zeros be simple.
   e) There be no missing terms in the numerator and the denominator polynomials unless all even or odd terms are missing.

2. Test for necessary and sufficient conditions:
   a) If P(s) be real when s real.
   b) If P(s)=N(s)/D(s), then N(s)+D(s) must be Hurwitz.

   This requires that:
   i) The continued partial fraction expansion of the Hurwitz test give only real and positive coefficients and
   ii) The continued fraction not end prematurely.
   c) In order that Real[P(jω)] ≥ 0 for all ω, it is necessary and sufficient that

      \[ A(ω^2) = m_1 m_2 - n_1 n_2 \]

      have no real positive roots and of odd multiplicity.

      \[ A(ω^2) \]

      Here, \( m_1 \) & \( m_2 \) are even parts of the numerator and the denominator polynomials and \( n_1 \) & \( n_2 \) odd parts.

3. Equivalent requirements: Condition A and any one in B.

   A. P(s) is real function when s is real. Equivalently:
      a) Arg[P(s)] = 0 or π when Arg[s] = 0.
      b) \( W = (P(s)-1)/(P(s)+1) \) is real when s is real.

   B. a) \[ \text{Real}[P(s)] ≥ 0 \] for Real[s] ≥ 0
      b) \[ \text{Real}[P(s)] ≥ 0 \] for Real[s] ≥ 0
      c) \( P(s) \) has no poles in the right half of s-plane.
      \[ \text{ii) Imaginary axis poles of } P(s) \text{ are simple; residues evaluated at these poles are real and positive.} \]
      \[ \text{iii) Real}[P(jω)] ≥ 0 \text{ for } 0 ≤ ω ≤ ∞ \]
      d) \[ W(s)=P(s)-1/P(s)+1 ≤ 1 \text{ for } \text{Real}[s] ≥ 0 \]
      e) \( W(s) \) has no poles on the imaginary axis or in the RHP of s-plane.
      \[ \text{ii) } |W(jω)| ≤ 1 \text{ for } 0 ≤ ω ≤ ∞ \]
      f) \( P(s) = n(s)/d(s), n(s)+d(s) \) is Hurwitz.
      \[ \text{ii) Real}[P(jω)] ≥ 0 \text{ for } 0 ≤ ω ≤ ∞ \]

Thus, any stable transfer function (with in a scaling factor) can be realized using a passive network built from those four elements. The modifier, with in a scaling factor, is needed because of power considerations associated with passive networks.

To summarize, the driving point functions of passive networks made of R, L, C and BT elements are restricted to rational functions known as PR functions. They can be either BIBO stable or only marginally stable. On the other hand, any stable LTI system response belonging to the class of constant coefficient linear differential equations involving the inputs and the outputs can be realized (to within a scale factor) as the transfer characteristic of a passive network.

6.2.4 Admittance / Impedance Matrices of Multi-port LTI Passive nets

In this section we will study the characteristics of admittance and impedance matrices of some special multi-port networks constructed from linear, time-invariant passive electrical elements.

6.2.4.1 Multi-port gyrators

In chapter # 5, we defined this multi-port network as an element in terms of its admittance matrix Y (a voltage controlled device) or the impedance matrix Z (a current controlled device). Y and Z are constant, skew symmetric (or antimetric) and hence positive semi-definite.

Though a gyrator is defined to be a basic building block that is also constrained to lossless, the actual implementation of gyrators involve transistors or operational amplifiers (building blocks of active circuits). Hence imperfections in the various elements used to build a gyrator can make it lossy or active and the gyrators unstable. In fact, such problems have limited the earlier use of gyrators in practical applications. In our case, we are more interested in a digital realization where we can maintain the skew symmetric property (which gives rise to the lossless characteristic) even under finite precision arithmetic.

6.2.4.2 Resistive N-port Networks & their corresponding Z/Y matrices

Next we consider For simplicity, we consider a two-port network as shown in Fig. 6.4 and extend the results to the general N-port network. Considering Fig. 6.4a, where we have shown a T equivalent circuit and it is known that any two-port network consisting of a number of resistors can be represented by its T
circuit equivalent. In the figure all the resistance values have to be positive. From the T circuit, we can write the v-i relationship for the two-port network in terms of the resistive matrix $R$ and the individual resistance values as:

$$
\begin{bmatrix}
    v_1(t) \\
    v_2(t)
\end{bmatrix} = \mathbf{v} = \mathbf{R} \mathbf{i} = \begin{bmatrix}
    R_{11} & R_{12} \\
    R_{21} & R_{22}
\end{bmatrix} \begin{bmatrix}
    i_1(t) \\
    i_2(t)
\end{bmatrix} = \begin{bmatrix}
    R_1 + \hat{R} \\
    \hat{R} + R_2
\end{bmatrix} \begin{bmatrix}
    i_1(t) \\
    i_2(t)
\end{bmatrix}
$$

(6.16)

Thus, we find:

$$R_{12} = R_{21} \geq 0, \quad R_{11} \geq R_{12} \quad \text{and} \quad R_{22} \geq R_{21}$$

(6.17)

That is $\mathbf{R}$ is positive definite or semi-definite with all terms positive. In fact, for the general N-port resistive network (with the resistive matrix $\mathbf{R}$), we can show that:

$$
\begin{align*}
R_{ij} &\geq 0, \\
R_{ji} &= R_{ij}, \\
R_{ii} &\geq \sum_{j=1, j\neq i}^{N} R_{ij}
\end{align*}
$$

(6.18)

That is, the impedance matrix of a N-port resistive network has only positive elements, is symmetric and positive definite or positive semi-definite.

Similarly, considering the \( \pi \) equivalent circuit shown in figure 6.4b with the elements denoted by the conductance values we can write the v-i relationship for the two-port network in terms of the conductance matrix $\mathbf{G}$ and the individual conductance values as:

$$
\begin{bmatrix}
    i_1(t) \\
    i_2(t)
\end{bmatrix} = \mathbf{i} = \mathbf{G} \mathbf{v} = \begin{bmatrix}
    G_{11} & G_{12} \\
    G_{21} & G_{22}
\end{bmatrix} \begin{bmatrix}
    v_1(t) \\
    v_2(t)
\end{bmatrix} = \begin{bmatrix}
    G_1 + \hat{G} \\
    -\hat{G} + G_2
\end{bmatrix} \begin{bmatrix}
    v_1(t) \\
    v_2(t)
\end{bmatrix}
$$

(6.19)

where

$$G_{12} = G_{21} \leq 0, \quad G_{11} \geq |G_{12}| \quad \text{and} \quad G_{22} \geq |G_{21}|
$$

(6.20)

That is $\mathbf{G}$ is symmetric with positive diagonal elements and non-positive off-diagonal elements and is positive definite or semi-definite. In fact, for the general N-port resistive network (with the resistive matrix $\mathbf{G}$), we can show that:

$$G_{ii} \geq 0 \quad \text{for} \quad i = 1 \quad \text{to} \quad N
$$

$$G_{ij} = G_{ji} \leq 0 \quad \text{for} \quad i, j = 1 \quad \text{to} \quad N \quad \text{and} \quad j \neq i
$$

(6.21)

$$G_{ii} \geq \sum_{j=1, j\neq i}^{N} |G_{ij}| \quad \text{for} \quad i = 1 \quad \text{to} \quad N
$$

That is, the admittance matrix of a N-port resistive network is symmetric with positive diagonal elements, negative off-diagonal elements, and positive definite or positive semi-definite.

6.2.4.3 Lossy static N-port Network formed from M-Port (M>N) Gyrator and passive Resistors & their corresponding R/G matrices

Let us consider a M-port (with $M = 2N$) linear gyrator terminated on its last N-ports and the individual resistance values as:

$$
\begin{bmatrix}
    i_{G1} \\
    i_{G2}
\end{bmatrix} = \begin{bmatrix}
    G_{G11} & G_{G12} \\
    -G_{G12} & G_{G22}
\end{bmatrix} \begin{bmatrix}
    v_{G1} \\
    v_{G2}
\end{bmatrix}
$$

(6.22)

where $i_{G1}$ and $v_{G1}$ are the vector of currents through and the vector of voltages across the first N-ports, similarly $i_{G2}$ and $v_{G2}$ vectors of currents and voltages corresponding to the last N-ports and $G_{Gij}$’s are sub-matrices dimensions N by N. For the last N-ports, due to the resistive terminations, we obtain the following relationship between the currents and voltages:
We can note that the conductance matrix has a special form. It is the sum of two matrices: One is an antimetric matrix; the other is a symmetric matrix and is positive definite or semi-definite. Thus, we find what is well known in passive network synthesis: Any constant matrix $\mathbf{A}$ where $\mathbf{A} + \mathbf{A}^\top$ is positive definite can be realized as the conductance or resistance matrix of a N-port passive network with resistors, transformers and gyrators.

### 6.2.4.4 The General LTI dynamics resulting from static, lossy N-port Networks terminated with dynamics Elements

Let us consider the static, lossy N-port network of Fig. 6.5 and terminate the N-ports with unit valued linear capacitors as shown in Fig. 6.6. Using KCL, we get:

\[
\begin{bmatrix}
\mathbf{I}_{G1} \\
\mathbf{I}_{G2}
\end{bmatrix} = \begin{bmatrix}
\mathbf{G}_{G1} & \mathbf{G}_{G2}
\end{bmatrix} \begin{bmatrix}
\mathbf{v}_{G1} \\
\mathbf{v}_{G2}
\end{bmatrix}
\]

where $\mathbf{G}$ is a diagonal matrix with the terminating conductance values and the negative signs occurs because of the assumed directions for the gyrator and the terminating resistance currents. Combining the expression for $\mathbf{I}_{G2}$ from (6.23) and (6.22), we get:

\[
\mathbf{v}_{G2} = [\mathbf{G} + \mathbf{G}_{G2}]^{-1} \mathbf{G}_{G12}^{-1} \mathbf{v}_{G1}
\]

and

\[
\begin{bmatrix}
\mathbf{I}_{G1} \\
\mathbf{I}_{G2}
\end{bmatrix} = \begin{bmatrix}
\mathbf{G}_{G1} & \mathbf{G}_{G2}
\end{bmatrix} \begin{bmatrix}
\mathbf{v}_{G1} \\
\mathbf{v}_{G2}
\end{bmatrix}
\]

Thus, we find what is well known in passive network synthesis: Any constant matrix $\mathbf{A}$ where $\mathbf{A} + \mathbf{A}^\top$ is positive definite can be realized as the conductance or resistance matrix of a N-port passive network with resistors, transformers and gyrators.

![Diagram 6-5](image)

Figure 6-5. A Systematic procedure for forming a N-port resistive network using a 2N-port lossless gyrator and terminating at its last N-ports passive resistors.

![Diagram 6-6](image)

Figure 6-6. A passive network architecture corresponding to a general N-th order LTI dynamics.
\[ i_c = Cv_c = -i_{LG1} + i_s \]  \hspace{1cm} (6.27)

where \( i_c \) is the vector of currents through the capacitors, \( i_{LG1} \) is the vector of currents into the N-ports of the lossy gyrator, \( i_s \) is the vector of source currents, \( v_c \) is the vector of capacitor voltages (equal to the gyrator voltages) and \( C \) is a diagonal matrix with the capacitor values. From (6.25) and the above expression, we find:

\[ Cv_c = -\hat{G}v_c + i_s = -G_{G11} + G_{G12}G_{22} + G_{G22}v_c + i_s \]  \hspace{1cm} (6.28)

The general form of the dynamics is obvious from the above expression.

**6.2.4.5 The General LTI dynamics resulting from static, lossy N-port Networks terminated with dynamics Elements Vs the General Dynamics**

The dynamics in expression (6.28) has been obtained by connecting both the dynamic elements (that give rise to state equations) and the sources in the same ports (similar to driving point characteristic when \( N=1 \)). Instead, let us now look at the dynamics when the sources and the reactive elements are connected to different set of ports as shown in Fig. 6.7. In the figure, we have one 2N-port lossy gyrator represented by its constant conductance matrix \( G_{LG} \), one \((N + M)\)-port lossless gyrator represented by its constant resistance matrix \( R_G \) (with a special form as given below to simply the expressions), \( N \) inductors, and \( M \)-current source\(^3\). From the individual elements’ v-i relationships and the interconnections shown in Fig. 6.7, we obtain the following expressions:

**Lossless gyrator:**

\[ v_G = \begin{bmatrix} v_{G1} \\ v_{G2} \end{bmatrix} = R_G i_G \]  \hspace{1cm} (6.29)

\[ i_{LG} = \begin{bmatrix} i_{LG1} \\ i_{LG2} \end{bmatrix} = G_{LG} v_{LG} \]  \hspace{1cm} (6.30)

where \( L \) is a diagonal matrix of inductance values (positive). From (6.29) to (6.31) we obtain:

\[ v_{G1} = v_{LG2} \]
\[ v_{G2} = v_L = L \frac{di_L}{dt} \]
\[ i_{LG1} = -i_s \]
\[ i_{LG2} = -i_L \]
\[ v_{LG1} = i_s \]  \hspace{1cm} (6.31)

\(^3\) We need to use current sources and inductors because of the representation chosen for the gyrators and to make the general form of the dynamics as complex as possible.

![Figure 6-7. A more complex passive architecture to generate any N-th order LTI dynamics.](image-url)
\[ \dot{i}_L = -L^{-1}R_{G12}(G^{\dagger}_{L02} - G_{L02}G^\dagger_{LG12}G_{LG12})R_{G12} + L^{-1}R_{G12}G_{L02}G^\dagger_{LG12}i_s \]

where we have identified the matrices that are positive semi-definite (but need not necessarily be symmetric). \( R_{G12} \) and \( G_{LG12} \) can be arbitrary except that the complete matrices \( R_G \) and \( G_L \) (that contain these sub-matrices) have to be positive semi-definite. Thus, the general form of the state space dynamics corresponding to lossy networks and the restrictions on the \( A \) and \( B \) matrices can be noted from this equation.

In mathematical control theory, as mentioned before we in general use a matrix expression of the form:

\[ \dot{x} = Ax + Bu \]

(6.33)

to represent an N-th order dynamics. It is well known that (Kalman-Yakubovic Theorem) the LTI system with the state matrix as given above is stable iff for any given positive definite matrix \( Q \), the solution of

\[ AP + PA^\dagger = -Q \]

(6.34)

leads to a matrix \( P \) that is also positive definite. That is, to check the stability of a given LTI system, we need to select \( Q \) first (and not \( P \)), solve for \( P \) from the above equation and check if the resulting \( P \) matrix is PD or not. If we select \( P \) first as a PD matrix (otherwise arbitrary) and solve for \( Q \), the resulting \( Q \) may not necessarily be PD even if the LTI system is stable.

Comparing (6.32), the general form of dynamics from lossy networks with the general state space model in equation (6.34), we find that:

\[ A = -L^{-1}R_{G12}^\dagger(G_{L02} - G_{L02}G^\dagger_{LG12}G_{LG12})R_{G12} \]

(6.35)

where the negative of the matrix on the right side of the equation is positive semi-definite. That is, the dynamics given in (6.32) derived from the network of figure 6.7 is not as general as the dynamics in (6.34). That is, we can find matrices \( A \) corresponding to stable dynamics and where the negative of the \( A \) matrix need not correspond to the conductance or the resistance or the hybrid matrix of a static lossy network. Of course, the difference comes because the requirement that the dynamics be stable constrains only the denominator of the transfer function (the polynomial, \( \det(sI - A) \)). On the other hand, a conductance or resistance or even a hybrid matrix carry in it additional information such as passivity and the interconnection properties. Thus, if the state space model in equation (6.34) corresponds to a physical (lossy) system, the property:

\[ \det(sI - A) \neq 0 \text{ for } \text{real}[s] \geq 0 \]

(6.36)

is just a necessary but not a sufficient one. (We should be able to decompose \( A \) as given by equation 6.35). Thus, representations such as:

\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
-\alpha & -\alpha & -\alpha & \cdots & -\alpha \\
\end{bmatrix}
\]

(6.37)

are abstract mathematical models that do not really represent passive (or lossy) systems.

6.2.5 Impedance Scaling and Frequency Transformations in LTI Passive Nets

In chapter 2, we studied that LTI property enables us to characterize LTI systems using one function, the impulse response. In this section, we will explain two important properties that go with LTI systems realized as passive networks.

6.2.5.1 Impedance Scaling

Consider the three linear passive elements, resistors, inductors and capacitors with values \( R, L \) and \( C \) respectively. Assuming no initial stored energy in the inductor and the capacitor, these elements can be expressed in the frequency domain by their driving point impedance functions as:

\[
\begin{align*}
Z_R(s) &= V_R(s) / I_R(s) = R \\
Z_L(s) &= V_L(s) / I_L(s) = Ls \\
Z_C(s) &= V_C(s) / I_C(s) = \frac{1}{cs}
\end{align*}
\]

(6.38)
Suppose the values of these elements are changed to \( \hat{R} = kR \), \( \hat{L} = kL \) and \( \hat{c} = \frac{c}{k} \) (k > 0). The new impedance functions are:

\[
\begin{align*}
\hat{Z}_R(s) &= \frac{V_R(s)}{I_R(s)} = \hat{R} = kR = kZ_R(s) \\
\hat{Z}_L(s) &= \frac{V_L(s)}{I_L(s)} = \hat{L} = kL = kZ_L(s) \\
\hat{Z}_C(s) &= \frac{V_C(s)}{I_C(s)} = \hat{c} = \frac{c}{k} = kZ_C(s)
\end{align*}
\]

That is, if we multiply the value of inductor and the resistor by a factor k (k > 0) and divide the value of the capacitor by the factor k, the impedance functions also change proportionally. Thus the process of changing the all elements values by the same factor k as defined here is known as impedance scaling. We may ask what would be the effect of impedance scaling on the driving point and transfer functions of complex passive networks made up of these three elements and the Brune transformer. As a specific example, let us consider a general two port network as shown in Fig 6.8a with a number of inductors, capacitors and resistors. We can obtain a new network by impedance scaling as shown in the figure 6.8b. Four I/O transformations \( \{v_1(t) \rightarrow i_1(t), \ v_2(t) \rightarrow i_2(t), \ i_1(t) \rightarrow v_1(t) \text{ and} \ i_1(t) \rightarrow v_2(t) \} \), some driving point transformations and others transfer functions, are possible for the two, two-port networks shown. It can be shown that:

\[
\begin{align*}
\frac{\hat{V}_1(s)}{I_1(s)} &= \hat{Z}_{11}(s) = k \frac{V_1(s)}{I_1(s)} = kZ_{11}(s) \\
\frac{i_1(s)}{V_1(s)} &= \hat{Y}_{11}(s) = \frac{1}{k} \frac{V_1(s)}{I_1(s)} = \frac{1}{k} Y_{11}(s) \\
\frac{\hat{V}_2(s)}{I_1(s)} &= \hat{Z}_{21}(s) = k \frac{V_2(s)}{I_1(s)} = kZ_{21}(s) \tag{6.40} \\
\frac{i_1(s)}{V_1(s)} &= \hat{Y}_{21}(s) = \frac{1}{k} \frac{V_2(s)}{I_1(s)} = \frac{1}{k} Y_{21}(s) \\
\frac{\hat{V}_2(s)}{V_1(s)} &= \hat{T}_{21}(s) = \frac{V_2(s)}{V_1(s)} = T_{21}(s)
\end{align*}
\]

That is, the process of scaling the element values (or impedance scaling) by a constant factor leads to a new network whose transfer and driving point functions with the dimension of impedance are scaled (by the same constant factor) versions of the corresponding transfer and driving point functions of the original network. On the other hand, the process of impedance scaling leaves transfer functions with no dimensions (voltage to voltage and current to current transformations) unchanged. In all the cases, the pole and zero locations remain unchanged.

We will discuss the importance of impedance scaling after we discuss the process of frequency scaling.

### 6.2.5.2 Frequency Scaling

Suppose we scale the values of the inductors and capacitors by a factor k (\( L_1 \) to \( \hat{L}_1 = kL_1 \) and \( \hat{c} = \frac{c}{k} \) to \( kC_1 \), k > 0) and leave the values of the resistors unchanged (Fig. 6.9). Using similar notations, we can show that:
That is, scaling the values of the inductors and capacitors by a factor \( k \) has the same effect as that of scaling the complex frequency variable \( s \) by the same scale factor \( k \). Thus if all the reactive elements are changed by a factor \( k \) in a complex network, the driving point functions and the transfer functions of the new network can be obtained from the corresponding driving point and transfer functions of the old network, and by replacing the complex frequency \( s \) by the new complex frequency variable \( k s \). That is,

\[
H_{\text{new}}(s) = H_{\text{old}}(ks)
\]

(6.42)

where \( H(s) \) stands for both driving point and transfer functions. It can be observed that the pole and zero locations will be changed accordingly by the factor "1/k".

In the time domain, the new impulse response will be given by:

\[
h_{\text{new}}(t) = \frac{1}{k} h_{\text{old}}(tk)
\]

(6.43)

Thus the response in the time domain changes by the same factor. The process of changing the reactive element values is known as frequency scaling. Impedance scaling allows us to work with simple element values (1 ohm etc.) that leads to transfer functions (and driving point functions) with simple coefficients and simplify the task of analyzing their responses. Though such element values may be impractical, through impedance scaling we can convert the network to a realizable one without changing the response. Similarly, frequency scaling allows us to work in the normalized frequency domain (cut off frequency, \( \omega_c = 1 \text{ rad/ sec} \) regardless of the application domain and scale the values as necessary. In practice, frequency scaling has to be applied first to bring the frequency response to the required frequencies, followed by impedance scaling.

**Example:** Consider the lattice network shown in figure 6.10a. The network has the current to voltage transfer function:

\[
z_{12}(s) = \frac{V_2(s)}{I_1(s)} = \frac{s^2 - s + 1}{s^2 + s + 1}
\]

(6.44)

The frequency response of this transfer function is:

\[
z_{12}(j\omega) = \frac{V_2(j\omega)}{I_1(j\omega)} = \frac{(j\omega)^2 - j\omega + 1}{(j\omega)^2 + j\omega + 1}
\]

(6.45a)

\[
|z_{12}(j\omega)| = \sqrt{\frac{(1-\omega^2)^2 + (-\omega)^2}{(1-\omega^2)^2 + (\omega)^2}} = 1
\]

(6.45b)

\[
\angle z_{12}(j\omega) = -2\tan^{-1}\left(\frac{\omega}{1-\omega^2}\right)
\]

(6.45c)

Note that the transfer function magnitude has a value of one for all values of the frequency. That is, input sinusoids of any frequency is passed with no gain or
attenuation. Such a transfer function is known as an all-pass transfer function.\(^4\) It can be noted that the zeros of such transfer functions are mirror images (with respect to the origin) of the poles. Of course, the phase response is a function of the frequency \(\omega\). The phase response starts at zero degree, reaches -180 degree at \(\omega = 1\) and approach -360 as \(\omega \to \infty\). Such networks (or transfer functions) are known as phase correction or delay equalization networks.

Let us say, now we want to move the frequency at which the phase reaches -180 to 100,000 rad/sec (from 1 rad/sec) and impedance scale the network such that the terminating resistance is 600 ohms. Frequency scaling by the factor 100,000 first changes the inductance values to \(10^{-5} \text{ H}\), the capacitance values to \(10^{-5} \text{ F}\) and leave the resistance value unchanged at 1 ohm. Impedance scaling by the factor 600 next changes the inductance values to 6.0 mH, the capacitance values to 0.0167 \(\mu\text{F}\) and the resistance value to 600 \(\Omega\). The resulting network is shown in figure 6.10b. Working with the normalized values thus help us avoiding dealing with such odd numbers.

\[ i_s = i_1 + i_2 \] for current controlled inductors \hspace{1cm} (6.46a)

or

\[ i_s = i_1(\phi_1) + i_2(\phi_2) \] for flux controlled inductors \hspace{1cm} (6.46b)

indicating that in both cases, the interconnections constrains one of the two, supposedly independent variables. In the case LTI elements such interconnection will just reduce the degree of the transfer function etc. However, in the case of nonlinear systems, such equalities may lead to indeterminate

6.3 Circuits made of Nonlinear, Time -Invariant Passive Elements

We can form nonlinear, time-invariant passive circuits from nonlinear, time-invariant passive elements that we have seen before. As in the case of linear, time-invariant passive circuit formation, care has to exercised when such a circuit is formed.

The basic circuit laws KCL and KVL and other concepts such as energy conservation hold for nonlinear circuits as well. Thus, loops consisting of capacitor only (or capacitors and independent voltage sources) and nodes consisting of inductors only (or inductors and independent current sources) have to be avoided. This restriction holds whether the inductor (capacitor) is charge (flux) controlled or voltage (current) controlled. This can be seen from the network example shown in fig. 6.11. Writing the KCL for node 1 we obtain:

\[ i_s = i_1 + i_2 \] for current controlled inductors \hspace{1cm} (6.46a)

or

\[ i_s = i_1(\phi_1) + i_2(\phi_2) \] for flux controlled inductors \hspace{1cm} (6.46b)

\[ i_s = i_1 + i_2 \] for current controlled inductors \hspace{1cm} (6.46a)

or

\[ i_s = i_1(\phi_1) + i_2(\phi_2) \] for flux controlled inductors \hspace{1cm} (6.46b)
solutions since we will be making the fluxes as the independent variables and the flux-to-current mappings can be many-to-one. Thus, such interconnections have to be avoided in practical applications.

A similar consideration would limit the types of nonlinear resistors placed in series with inductors (linear and nonlinear) or in parallel with capacitors (linear or nonlinear). An example network is shown in fig. 6.12, where, we have a circuit with two linear capacitor, two linear inductor and two nonlinear resistors, \( R_1 \) and \( R_2 \). From the network, we can write 4 state-space equations as:

\[
\begin{align*}
\dot{i}_{L1} &= v_s - v_{c1} - v_{R1} \\
\dot{v}_{c1} &= i_{L1} - i_{L2} \\
\dot{i}_{L2} &= v_{c1} - v_{c2} \\
\dot{v}_{c2} &= i_{L2} - i_{R1}
\end{align*}
\]

(6.47)

and two algebraic equations

\[
\begin{align*}
i_{L1} &= i_{R1} \\
v_{c1} &= v_{R2}
\end{align*}
\]

(6.48)

where the possibilities for the nonlinear resistors are:

If we choose the nonlinear resistor \( R_1 \) to be current controlled and the nonlinear resistor \( R_2 \) to be voltage controlled, we can solve the four state equations in (6.47) that also satisfies the two algebraic equations in (6.48). This will be true regardless of whether the nonlinear characteristics are single valued or multi-valued.

However, if we choose the nonlinear resistor \( R_1 \) to be voltage controlled and or the nonlinear resistor \( R_2 \) to be current controlled, the solvability of the 4-state equations subject to the constraints (given by the two algebraic equations and the v-i relationship of the two nonlinear resistive elements) may not be possible. Hence such interconnections also should be avoided.

6.3.1 Transient and forced response of nonlinear TI Passive circuits

We will look at a number of examples of nonlinear, TI passive circuits and their transient and forced responses. We can arrive at useful conclusions about the responses from these simple examples.

Example 1: We show one such circuit in the fig. 6.13. The network consists of a flux controlled inductor (with just one relaxation point \( \phi_r = 0 \)) and a current controlled resistor with the I/O relationships as:

\[
\begin{align*}
i_{L}\{\phi\} &= a_L\phi(1+b_L \sin[c_L \phi]) \\
v_r\{\phi\} &= a_r i_r(1+b_r \sin[c_r i_r])
\end{align*}
\]

(6.50)

where \( a_L \) and \( a_r \) have to be positive and \( |b_L| \) and \( |b_r| < 1 \) for the elements to be passive. As shown in the figure, both the element characteristics are one to many mappings (small scale inverse inductance and small scale inverse conductance becomes zero as well as negative). Let the flux at time \( t = 0 \) be denoted as \( \phi(0) \neq 0 \) and the two terminals of the elements are connected together. We can ask how \( \phi(t) \) and other variables will evolve as a function of time? That is, what will be the transient response of the network?

The inductance has \( \phi_r = 0 \) as the relaxation point. Hence there is initial stored energy in the inductor for \( \phi(0) \neq 0 \) given by the area under the curve \( i_{L}\{\phi\} \) and \( \phi \) axis. A passive resistance that always consumes power is connected to this element. There is no other source to supply additional power or energy. Therefore it is clear that \( \phi(t) \), \( i_{L}(t) \), \( v_{R}(t) \) will go to zero as \( t \to \infty \). Further, \( |\phi(t)| \) will be monotonically decreasing as the energy is continuously
depleted. However, the decay will not be uniform as the current and the voltage across the resistor will vary tremendously and hence the power consumed by the resistor would vary from almost zero to quite large as \( \phi(t) \) changes.

The governing equations for the first order nonlinear network are:

\[
i_R(t) = i_L(t) = i_L[\phi(t)] \\
\dot{\phi}(t) = -v_R(t) = -v_R[i_R(t)]
\]  
(6.51a)

or

\[
\dot{\phi}(t) = -v_R[i_L[\phi(t)]] = -\dot{v}_R[\phi(t)]
\]  
(6.51b)

where \( \dot{v}_R[\phi(t)] \) is a mapping that combines the two mappings \( i_L[\phi(t)] \) and \( v_R[i_R(t)] \) into a single one.

A numerical solution of these equations for some values of \( \phi(0) \) are shown in fig. 6.14. As expected the flux, \( \phi(t) \), and other responses goes to zero and the waveforms are more complex as compared to the waveforms from a first order linear network.

Of course, it is very easy to prove the stability of the dynamics. The real Lyapunov function (not just a candidate) is the energy stored in the reactive element and is given by:

\[
E[\phi] = 0.5a_L \phi^2 + a_L b_L \left( \sin[c_L \phi] - c_L \phi \cos[c_L \phi] \right)/c_L^2
\]  
(6.52a)

which is a complex function of the state variable \( \phi \) and not in the simple form of a quadratic equation. The derivative of this LF along the system trajectory is negative of the power consumed by the passive resistor, and given by:

\[
\frac{dE}{dt}_{\text{trajectory}} = -p_r(t)
\]  
(6.52b)

\[
= -a_r a_L \phi (1 + b_L \sin[c_L \phi])(1 + b_r \sin[c_a \phi](1 + b_L \sin[c_L \phi])])
\]

which will always be negative when the nonlinear resistor is passive. The power of the building block approach should be obvious from this first-order dynamics. Without the network interpretation, it would have been very difficult to arrive at the proper LF even for this simple example.

---

\(^5\) Recall that the transient response of a first order linear Time-invariant network is proportional to \( e^{-\omega t} \).
Example 2  Let us consider a first order network with a nonlinear resistor and a nonlinear capacitor as shown in Fig. 6.15a. The nonlinear resistor characteristic is assumed to be similar to the one in example 1 but here we need to use a voltage controlled resistor. That is:

\[ i_r[v_r] = a_r v_r (1 + b_r \sin[c_r v_r]) \]  

(6.53)

The nonlinear capacitor is characterized by the \( v_c - q \) relationship:

\[ v_c[q] = q(q^2 - 9)(q^2 - 16) = q(q^4 - 25q^2 + 144) \]  

(6.54)

and is shown graphically in Fig. 6.15b. Note that the voltage becomes zero for \( q = 0 \), \( q = \pm 3 \) and \( q = \pm 4 \). From the \( v_c - q \) curve, we can note that only \( q = 0 \) qualifies to be the relaxation point. We can calculate the energy stored in the capacitor as a function of the charge \( q \) as:
The energy expression is also plotted in figure 6.15b. We can make some interesting observations from the energy plot. The energy stored in the capacitance is greater than or equal to zero indicating that the chosen expression for $v_c - q$ represents a valid capacitor. The energy corresponding to $q = 0$ is zero indicating that it is a relaxation point. We also note that the energy curve is not monotonically increasing. In fact, we find a local maxima at $q = \pm 3$ and local minima at $q = 4$. Thus, we can expect some unusual response from this simple (in terms of the filter order) circuit.

The dynamics of the network is given by:

$$\dot{q} = i_c = -i_r[v_r] = -i_r[v_c]$$

(6.56)

In Fig. 6.15c, we show the response $i_c(t)$ for two initial values for $q(0)$. The response corresponding to $q(0) = 2.7$ moves to $q = 0$, the relaxation point for the capacitor. However, the response corresponding to $q(0) = 3.3$ moves to $q = 4$ not a relaxation point but the value corresponding to local minima of the energy curve. In fact, for all values of $q(0)$ where $0 < q(0) < 3$ will move towards $q = 0$ where as the values $3 < q(0) < \infty$ will lead to $q = 4$ as shown in the figure. Thus, all the energy minimas (not just global) become the attractive points for the dynamics and the circuit.

**Example 3** Consider a network consisting of a linear inductor of value $L$ henry, a nonlinear capacitor and a nonlinear resistor connected in parallel (Fig. 6.16a). The describing equations are:

$$i_L[\phi] = \frac{1}{L}[\phi]$$

$$v_c[q] = q(q^2 - 9)(q^2 - 16)$$

(6.57a)

$$i_r[v_c] = a_i v_r (1 + b_r \sin[c_r v_r])$$

Figure 6-15. a) A first-order network with a nonlinear capacitor and a nonlinear resistor; b) The characteristics of the nonlinear capacitor. The charge-voltage waveform goes into the second- and the fourth-quadrant as well, making the stored energy waveform to have three minimas, one global, and two local. c) & d) Various responses as a function of time. The waveforms are highly complex due to the nature of the elements. The state reaches one of the three values, -4, 0, & 4, the values at which stored energy is minimum depending upon the initial conditions.
Figure 6-15 (Contd.)

Figure 6-16. a) A second-order network using the nonlinear capacitor and resistor of figure 6.15 and a linear inductor; b) Various responses for one initial condition; c) Phase portrait of the dynamics.
where the parameters are constrained so that the elements are passive. The resulting dynamics is given by:

\[
\dot{q} - \dot{\phi} = 0 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_c(q) \\ i_r(q) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_c(q) \tag{6.57b}
\]

In Fig. 6.16b, we show the various responses as a function of time. The obtained waveforms are complex with sudden phase reversal etc. The responses become zero (at least for the element parameters and the initial conditions used) as time progresses. We also show the phase plane plot \(q(t) - \phi(t)\) in Fig. 6.16c.

The LF for this dynamics is given by the energy stored in the two reactive elements, and is given by:

\[
E[q, \phi] = \frac{0.5\phi^2}{L} + q^2(72 - 6.25q^2 + 0.1667q^4) \tag{6.58a}
\]

which is positive for any values of the state variables and has minimas at \([0, 0], [4, 0]\) and \([-4, 0]\). Its derivative along the system trajectory is the negative of the power consumed by the resistor and is given by:

\[
\left.\frac{dE}{dt}\right|_{\text{sys traj}} = -p_r(t) = -a_c v_c(q)(1 + b_c \sin[c_c v_c(q)]) \tag{6.58b}
\]

which is always negative.

**Example 4** Next we consider a lossless circuit consisting of a nonlinear inductor and a nonlinear capacitor described by:

\[
\begin{align*}
i_L(\phi) &= a_L \phi(1 + b_L \sin[c_L \phi]) \\
v_c(q) &= a_c q(1 + b_c \cos[c_c q])
\end{align*} \tag{6.59}
\]

That is, the capacitor characteristic is made antimetric as shown in Fig. 6.17a. The various responses are shown in figures b and c. We can see complex oscillations that depend on the initial conditions. Note that this is different from what we saw in the case of the Van der Pol equation leading to the limit cycle behavior. The difference can be explained easily using the building block approach. The dynamics here correspond to that of a lossless (though nonlinear) circuit. Different initial conditions imply different amount of initial stored energy which is preserved due to the lossless nature of the circuit. Hence, we see oscillations whose amplitude depends on the initial conditions. In the case of Van der Pol dynamics, the equivalent circuit becomes complex with negative resistors as we will see in the next example.

Finally, note the phase plane plot in terms of the transformed state variables, \(v_c(q)\) and \(i_L(\phi)\) which looks so exotic. This plot should illustrate the power of nonlinear dynamics in generating complex and jazzy plots.

In Fig. 6.18, we show the responses of a lossy network formed by connecting in parallel a nonlinear resistor of the form used in example 3 to this lossless network. As can be suspected, the responses eventually die out due to the lossy nature of the circuit. The phase plane plot of the transformed state variables still looks strange due to the complex nature of the element characteristics used.

**Example 5** Van der Pol Equation

We have already come across the Van der Pol equation given by:

\[
m\ddot{x}(t) + 2c(x^2(t) - 1)\dot{x}(t) + kx(t) = 0 \tag{6.60a}
\]

where the parameters \(m, c, k\) are assumed to be positive. The system has one equilibrium point, \([x_e, \dot{x}_e]^T = [0, 0]^T\). By integrating the Van der Pol dynamics with respect to the independent variable, \(t\), we get:

\[
m \dot{x}(t) + 2c(t)\frac{x^2(t)}{3} - 1 + k \int_0^t x(t)dt = m \dot{x}(0) + 2c(0)\left[1 - \frac{x^2(0)}{3}\right] \tag{6.60b}
\]
Figure 6-17. Response of a lossless, nonlinear network. The inductor is nonlinear with a modulated flux-current characteristic similar to one used in figure 6.13, and the capacitor characteristic is as shown in figure a [nonlinear and antimetric]; b) The various responses (the two state variables, flux & charge, the transformed variables, the inductor current and the capacitor voltage) for some initial conditions; c) Phase plane plot (flux Vs charge and inductor current Vs capacitor voltage) of the dynamics.

Figure 6-18. Response of a lossy, 2nd-order nonlinear network with three nonlinear elements; a) The various responses (the two state variables, flux & charge, and the transformed variables, inductor current, & the capacitor charge) for some initial conditions; b) Phase portrait (flux Vs charge and inductor current Vs capacitor voltage) of the dynamics.
Thus, by equating $x(t)$ to the current in a series circuit, we obtain an equivalent circuit for the Van der Pol equation as shown in Fig. 6.19a, where we have two linear reactive elements, an independent voltage source, and a nonlinear resistor. The voltage-current relationship of the resistor is given by:

$$v_R[i_R] = 2c \left( \frac{i_R^2}{3} - 1 \right)$$

The resistor becomes a negative resistor (an controlled active source) for small values of the current magnitude and passive for large values can be noted. This complex resistor leads to an oscillatory response which is independent of the initial conditions.

### Example 6
Consider the simplified model of an underwater vehicle given below:

$$\dot{v}(t) + v(t)v(t) = u(t) \quad (6.61)$$

where $u(t)$ is the input thrust and $v(t)$ is the vehicle velocity. By equating the velocity to a current and the thrust to a voltage source, we obtain an equivalent electrical representation as shown in Fig. 6.20a. The equivalent network corresponds to a passive nonlinear network confirming that the model is a valid one, though perhaps a very simplified one. In Fig. 6.20b, we show the response of this network (or model) for some constant values for the thrust. From the figure, we can note that the DC response is not linear and the settling time varies depending on whether the thrust is increased (from zero to constant value) or decreased (from some constant value to zero). The response of the network as we reduce the thrust from a constant to zero is the transient response and it is not uniform if we consider the two regions, $0 < v(t) < 1$ and $1 < v(t) < \infty$. The energy left in the system is the energy left in the linear inductor and is $\frac{1}{2} v^2(t)$, and the power consumed by the resistance is given by $v^2(t)v(t)$. Thus, for $1 < v(t) < \infty$, $v(t)$ starts falling relatively fast and dies out rather slowly for $0 < v(t) < 1$. In fact, the later implies that the equilibrium point is not exponentially stable. Thus, using the network analogy and considerations such as power and energy, we can come to useful conclusions. In this example, we may ask if the term $v(t)v(t)$ used to represent the damping is sufficient or should we include another term proportional to $v(t)$ to the damping.

### Example 7
Let’s consider a first order network with one nonlinear inductor and a nonlinear resistor as shown in Fig. 6.21a. The characteristics of the elements are chosen as:

$$i_L[\phi] = a_L \tanh[b_L \phi]$$

$$v_r[i_r] = a_r \tanh[b_r i_r] \quad (6.62a)$$
with all the parameters positive. The dynamics of the network is given by:

\[
\dot{\phi}(t) + v_r[i_L(\phi(t))] = v_s(t) \quad (6.62b)
\]

We can note that the maximum voltage amplitude that can appear across the resistor is limited to \(a_r\) and a nonlinear inductor becomes a short for DC similar to a linear inductor. Thus, we need to be careful if we connect a DC voltage source to this network. The response of this network for two sinusoidal excitations \(A \sin(\omega t)\), \(A = 0.5, 5.0\), are shown in 6.21b. We can observe that the output is not sinusoidal because of the nonlinear nature of the elements. Also, when the peak to peak value is higher, the distortion becomes much more because of the restriction on the voltage that can appear across the resistor.

**Example 8** Let us consider the network of example 1 with the element characteristics given by:

\[
i_L[\phi] = a_L \phi (1 + b_L \sin[c_L \phi])
\]

\[
v_r[i_r] = a_r i_r (1 + b_r \sin[c_r i_r])
\]

and look at its DC response (Fig. 6.22a). The dynamics is given by:

\[
\dot{\phi}(t) + v_r[i_r] = v_s(t) \quad (6.64)
\]

where the two nonlinearities can be combined into a single nonlinearity given by:

\[
v_r[i_r] = a_r (a_L \phi (1 + b_L \sin[c_L \phi])) (1 + b_r \sin[c_r \phi]) \quad (6.65)
\]

and shown in Fig. 6.22b. The response for two initial conditions with the same DC input is shown in 6.22c. We can note that the steady state response depends on the initial condition as well. This is due to the many to one mapping characteristics that we have chosen for the elements. Thus, we can ask if such exotic characteristics are needed for real-world applications. The answer is, perhaps, we should restrict to monotonic characteristics while modeling real-world systems and consider non-monotonic characteristics if our aim is to build such complex systems. Exotic neural architectures can come under the latter category as we will discuss in the chapter on chapter.
6.3.2 Is Separation into transient and forced response really necessary?

In this section, we will discuss why a distinction between transient and forced response of nonlinear networks is necessary from a physical perspective.

We indicated in the last chapter (while discussing Lyapunov direct approach for stability analysis of nonlinear dynamics) that the conventional wisdom among researchers with an analytical bent is that the stability of nonlinear dynamics given either initial state values or constant forcing functions can be treated under the same umbrella. The rationale for this approach is that a nonlinear dynamics with constant forcing functions denoted as:

$$\dot{x} = f(x, t, u_{DC})$$

with the equilibrium point $x^*$ can be converted to an equivalent dynamics without any forcing functions as:

$$\dot{x} = f(\dot{x}, t)$$

where $\dot{x} = x - x^*$. Though this argument is perfectly valid from a mathematical viewpoint, it can alter the property of the network; i.e., a passive network with source may on transformation become a network with negative elements (non-passive network). We will explain this using two examples. In the first example, the transformation leads to a passive network; in example 2, it leads to a non-passive network.

**Example 1** Consider a nonlinear network consisting of series connection of a passive resistor, inductor and a capacitor, all assumed to be nonlinear. Let us also assume the characteristics of the inductor to be confined to the first- and the third-quadrant implying that it has only one relaxation point, the origin, and the stored energy is monotonically increasing. We will define the characteristics of the capacitor later. If we connect this network to a constant voltage source as shown in fig. 6.23a, we will obtain a second-order dynamics as:

\[
\begin{bmatrix}
q \\
\phi
\end{bmatrix}' = 
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix}
v_L[q] \\
v_R[i_L]\phi
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
v_{DC}
\end{bmatrix}
\]  

(6.68)

Depending upon the characteristics of the capacitor, the network will have one or more equilibrium points:

\[
[ q_{equ} \quad \phi_{equ} ]' = [ q^* \neq 0 \quad \phi^* = 0 ]
\]  

(6.69a)

and other signals in the network will take the values: Figure 6-22. a) A first-order network with a constant (DC) input (+1 and -1). b) The two nonlinearities combined into a single mapping of the state variable. c) The responses for two different initial conditions. The stable DC response depends on the initial conditions unlike the case of LTI networks. d) Phase portrait of the response.
\[ v' = v[e_q + q'] = v_{dc} \]
\[ i'_L = i_L[\phi'] = 0 \]
\[ v_i = v_i[\phi'] = 0 \]  \hspace{1cm} (6.69b)

Now using the change of variables:
\[ e_q = q - q^* \]
\[ e_L = \phi - \phi^* = \phi \]  \hspace{1cm} (6.70)

the dynamics changes to:
\[
\begin{bmatrix}
\dot{\hat{v}_c} \\
\dot{\hat{e}_e}
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{v}_c[e_q] + v_c[q'] \\
\hat{v}_c[e_q]
\end{bmatrix} - 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\hat{v}_c[e_q] + v_c[q'] \\
\hat{v}_c[e_q]
\end{bmatrix} - 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  \hspace{1cm} (6.71)

where \( \hat{v}_c[e_q] \) is a new function of \( e_q \) such that:
\[ v_c[e_q + q'] = \hat{v}_c[e_q] + v_c[q'] \text{ and } \hat{v}_c[0] = 0 \]  \hspace{1cm} (6.72)

Thus, as expected, the new dynamics has no forcing function and the origin is its equilibrium point. Therefore from a mathematical sense, we can analyze the stability of the dynamics in (6.68) driven by a constant source by analyzing the stability of the equilibrium point of the new dynamics (6.71) which has no forcing function. Thus, mathematically, the definition for equilibrium point, its stability etc. admit constant forcing functions as well in the dynamics.

Since the new dynamics vary only from the mapping of the capacitor characteristics, let us consider the various possibilities as shown in fig. 6.23. In fig. 6.23b, we show a \( v_c - q \) characteristics that is monotonically increasing. In this case, the dynamics has only one equilibrium point. The new mapping \( \hat{v}_c - \hat{q} = e_q \) is also monotonically increasing and hence point to a valid capacitor. Therefore, the network corresponding to the new dynamics is also passive and we can conclude that the equilibrium of the new dynamics will be absolutely stable.

In fig. 6.23c and 6.23d, we consider the cases where the \( v_c - q \) characteristics are non-monotonic (In fig. 6.23c, the waveform is confined to first and third quadrants, whereas as in fig. 6.23d, the waveform strays into the fourth quadrant). As can be seen from the figures, for some positive values of the voltage source \( v_{dc} \), we can have three solutions \( q = q_1, q_2, q_3 \) as \( q^* \). We can also show that \( q = q_1 \) and \( q_3 \) are the two stable solutions. In either case, the transformed waveform \( \hat{v}_c[e_q] \) points to a valid \( v_c - q \) curve of a capacitor with either the origin as the relaxation point and the other point as the point of local minima for the stored energy or vice versa. Therefore, the transformation leads to a new network that is still passive.

**Example 2**  \hspace{1cm} We will now look at another network formed from two linear capacitors, a linear resistor and a nonlinear gyrator and driven by a constant current source as shown in Fig. 6.24a. The dynamics of this network is:
\[
\begin{bmatrix}
\dot{v}_{c1} \\
\dot{v}_{c2}
\end{bmatrix} = 
\begin{bmatrix}
0 & v_{c1} \\
-v_{c1} & -1
\end{bmatrix}
\begin{bmatrix}
v_{c1} \\
v_{c2}
\end{bmatrix} + 
\begin{bmatrix}
0 \\
i_{dc}
\end{bmatrix}
\]  \hspace{1cm} (6.73)

This dynamics has one solution \( \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} = \begin{bmatrix} 0 \\ i_{dc} \end{bmatrix} \) when \( i_{dc} \) is negative. The response of the dynamics for one value of \( i_{dc} = -1.5 \) for a number of initial conditions is shown in figure 6.24b. We can see that the solution is stable. We can in fact show the stability of the solution by rewriting the dynamics in (6.73) as:
\[
\begin{bmatrix}
\dot{v}_{c1} \\
\dot{v}_{c2}
\end{bmatrix} = 
\begin{bmatrix}
i_{dc} & v_{c1} \\
-v_{c1} & -1
\end{bmatrix}
\begin{bmatrix}
v_{c1} \\
v_{c2} - i_{dc}
\end{bmatrix}
\]  \hspace{1cm} (6.74)

which points to a passive architecture\(^6\) as shown in figure 6.24c.

When \( i_{dc} \) is positive, the dynamics in (6.73) will have three solutions \( \left[ \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ i_{dc} \end{bmatrix} \) or \( \begin{bmatrix} \sqrt{i_{dc}} \\ 0 \end{bmatrix} \) or \( \begin{bmatrix} -\sqrt{i_{dc}} \\ 0 \end{bmatrix} \) of which the two, \( \begin{bmatrix} \sqrt{i_{dc}} \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} -\sqrt{i_{dc}} \\ 0 \end{bmatrix} \), are stable and one of which is reached based on the initial conditions (figure 6.24d). We can apply the change of variable \( \hat{v}_{c1} = v_{c1} - \sqrt{i_{dc}} \) (\( v_{c2} \) remains unchanged) to study the stability of the point \( \begin{bmatrix} \sqrt{i_{dc}} \\ 0 \end{bmatrix} \). The new dynamics is given by:

\(^6\) We noted in chapter 5 that a series connection of a capacitor and an ideal constant voltage source behaves like a lossless unit.
The transformation \( q_1 \) points to \( q_1 \) with \( q_1 = \hat{q} \) and \( q_1 = q - q_3 \) points to valid capacitor with either the origin as the relaxation point and the other point as the local minima for the stored energy or vice versa. That is, the capacitor characteristic becomes more complex than the one we started with. The non-monotonic capacitor characteristic that strays into the second-quadrant too. Again, the transformation \( \hat{V}_c [e_q] \) with \( e_q = q - q_1 \) or \( e_q = q - q_3 \) points to valid capacitor with either the origin as the relaxation point and the other point as the local minima for the stored energy or vice versa.

Figure 6-23. a) A second-order nonlinear passive net driven by a constant valued voltage source; b) The characteristic of the capacitor is assumed to be monotonically increasing (\( q=0 \) is the relaxation point and there is no other minima for the stored energy). The network/dynamics can be reduced to one with no source. The transformed characteristics \( \hat{V}_c vs q \) still point to a valid capacitor; c) The capacitor characteristic is assumed to be non-monotonic, but confined to first- and third-quadrants (still, \( q=0 \) is the relaxation point and there is no other minima for the stored energy). There are three possible equilibrium points (\( q_1, q_2, q_3 \)) of which \( q_1 \) and \( q_3 \) are stable. The transformation \( \hat{V}_c [e_q] \) with \( e_q = q - q_1 \) and \( e_q = q - q_3 \) points to valid capacitor with either the origin as the relaxation point and the other point as the local minima for the stored energy or vice versa. That is, the capacitor characteristic becomes more complex than the one we started with. 

\[
\begin{bmatrix}
\hat{v}_{c1} \\
\hat{v}_{c2}
\end{bmatrix} =
\begin{bmatrix}
0 & -\sqrt{\text{DC}} \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{v}_{c1} + \sqrt{\text{DC}}/2 \\
\hat{v}_{c2}
\end{bmatrix} +
\begin{bmatrix}
0 \\
-\sqrt{\text{DC}}/2
\end{bmatrix}
\]

(6.75)

This dynamics corresponds to a network with one or more negative resistors as shown in fig. 6.24e and no other network representation with only passive elements is possible. Transformation to bring the other stable equilibrium point to the origin will also lead to a dynamics that corresponds to a non-passive network. That is, in this case, the analytical approach leads to a new network which lacks an important property of the original network, that of passivity.

This last example illustrate the following:

Nonlinear passive \( \Rightarrow \) Dynamics with equilibrium point, the origin, net with no source \( \Leftrightarrow \) which is locally absolutely stable.

One may argue that such a scenario arises because we considered a dynamics having more than one equilibrium point which is true. However, we would like to point out two important points which have been overlooked by the analytically-inclined research community. First, the concept of an equilibrium point of a system points to the state of a system with no more stored energy (or no more energy that can be given away, as in the case of dynamics from reactive elements having only one relaxation point but with characteristics that move into second and fourth quadrants) and as we speculated in the previous chapter, Lyapunov function seems to have originated by considering if an initially stored energy in a system will be eventually dissipated if there is no other source to keep adding additional energy. However, in the case when a additional (constant) source is involved, we no longer have the scenario of stored energy being depleted. Here we have a situation where under equilibrium, constant power is being supplied by the sources continuously which gets consumed by the passive resistors. Thus, we rather have a power balance situation and not stored energy depletion. In fact, the network research community have recognized this some thirty years ago and showed that when a stable solution exists under constant source excitation, it occurs at the minima of a function known as mixed potential with the dimension of power. We will explain this function and the basic concepts in the following section.
Figure 6-24. a) A nonlinear passive net driven by a constant current source $i_Dc$; b) Phase portrait when $i_Dc$ is negative; c) Network obtained by a change of variables to move the equilibrium point to the origin. The resulting net is passive; d) Phase portrait when $i_Dc$ is positive. We have two stable equilibrium points; e) Network obtained by a change of variables to move the equilibrium point to the origin. The resulting net is non-passive.
6.3.2.1 DC Response of Dynamic Systems & the Function that gets minimized

The discussion here is limited to two terminal (one-port) linear and nonlinear devices. We consider a directed network with b branches (elements), n nodes, r inductors, s capacitors and (b-r-s) resistive elements and DC power sources. Since we have DC power sources in addition to passive or non-passive elements, for the sake of uniformity, we make the assumption that the current in each element enters at the negative voltage terminal and leaves through the positive voltage terminal as shown for the case of a nonlinear resistor in Fig. 6.25a (the true current will flow against the specified direction for the case of passive elements and hence a negative sign has to be added). The set of branch currents \( i = [i_1, i_2, \ldots, i_b] \) and the set of branch voltages \( v = [v_1, v_2, \ldots, v_b] \) are vectors in the b-dimensional Euclidean vector space \( \mathcal{E}_b \). Let \( \mathcal{I} \) be the set of all vectors in \( \mathcal{E}_b \) such that if \( x \in \mathcal{I} \) and the components of \( x \) are taken as the branch currents of the directed network, then KCL, \( \sum_{\text{node}} x_\mu = 0 \) must hold at every node. Similarly, we let \( v \) denote the set of all vectors in \( \mathcal{E}_b \) such that if \( x \in \mathcal{I} \) and the components of \( x \) are taken as the branch voltages, then KVL, \( \sum_{\text{loop}} x_\mu = 0 \) should be satisfied for every loop. Note that \( \mathcal{I} \) and \( v \) are orthogonal subspaces of \( \mathcal{E}_b \) since they are defined through linear equations.

**Theorem 1 (Tellegen's Theorem):** If \( i \in \mathcal{I} \) and \( v \in \mathcal{I} \), then the inner product \( (i, v) = 0 \), i.e., \( \mathcal{I} \) and \( \mathcal{V} \) are orthogonal subspaces of \( \mathcal{E}_b \).

**Theorem 2:** Let \( \Gamma \) denote a one-dimensional curve in \( \mathcal{E}_b \) with projections on \( \mathcal{I} \) and \( \mathcal{V} \) denoted by \( i \) and \( v \), respectively. That is, \( \Gamma \) is the solution trajectory of the network dynamics. Then, we can show using Tellegen's theorem that:

\[
\int_{t=1}^{b} \sum_{\mu=1}^{b} v_\mu di_\mu = \int_{t=1}^{b} \sum_{\mu=1}^{b} i_\mu dv_\mu = 0 \quad (6.76)
\]

Using the above expression, we can derive a function \( P \) of the state variables as follows. We consider the first summation in the above expression and split it into three summations corresponding to the inductors, capacitors and the rest of the elements as:

\[
\int_{t=1}^{b} \sum_{\mu=1}^{b} v_\mu di_\mu = \int_{t=1}^{b} \sum_{\mu=1}^{r} v_\mu di_\mu + \int_{t=1}^{r} \sum_{\mu=1}^{s} v_\mu di_\mu + \int_{t=1}^{s} \sum_{\mu=1}^{b} v_\mu di_\mu = 0 \quad (6.77)
\]

Since the voltage is the independent variable for the capacitors, let us integrate the second line-integral by parts. That is:

\[
\int_{t=1}^{b} \sum_{\mu=1}^{b} v_\mu di_\mu = \sum_{\mu=1}^{b} v_\mu i_\mu - \int_{t=1}^{b} \sum_{\mu=1}^{s} v_\mu dv_\mu
\]

and substitute in (6.79) to obtain:

\[
\int_{t=1}^{b} \sum_{\mu=1}^{b} v_\mu di_\mu + \sum_{\mu=1}^{b} v_\mu i_\mu - \int_{t=1}^{b} \sum_{\mu=1}^{s} v_\mu dv_\mu + \int_{t=1}^{b} \sum_{\mu=1}^{b} v_\mu di_\mu = 0
\]

where \( P \) is given by:

\[
P = \int_{t=1}^{b} \sum_{\mu=1}^{r} v_\mu dv_\mu + \sum_{\mu=1}^{s} v_\mu i_\mu \quad (6.80)
\]

Note that \( P \) depends only on the end points of \( \Gamma \). Thus, \( P \) is a function of the variable end point of \( \Gamma \) which, in turn, depends on the state variables \( i_L = [i_1, i_2, \ldots, i_r] \) (inductor currents) and \( v_c = [v_{r+1}, v_{r+2}, \ldots, v_{r+s}] \) (capacitor currents), i.e., \( P = P[i_L, v_c] \). It is also only defined up to a constant which depends on the choice of the fixed initial point of \( \Gamma \). Differentiating (6.79) with respect to the inductor currents and capacitor voltages, we obtain:

\[
\frac{d\phi_{li}}{dt} = -\frac{\partial P[i_L, v_c]}{\partial i_{li}} \quad \text{for } i = 1 \text{ to } r \quad (6.81a)
\]

and

\[
\frac{dq_{ci}}{dt} = \frac{\partial P[i_L, v_c]}{\partial v_{ci}} \quad \text{for } i = 1 \text{ to } s \quad (6.81b)
\]

\[7\] The theory given here also applies when the network has mutual inductance or any other multi-port (two-ports or higher) devices with a symmetric matrix. It is not applicable when we have multi-port (greater than two) devices with non-symmetric matrices (example: multi-port gyrators) and has to be modified.

\[8\] We write the inductor variables first followed by the capacitor variables and finally, the non-reactive element variables.

\[9\] Or, transformations of the state variables, if the inductors and capacitors are assumed respectively to be flux and charge controlled.
That is, the partial derivative of the function \( P[i_L, v_c] \) with respect to the inductor currents and capacitor voltages gives rise to the network dynamical equations. The minima of this function leads to the equilibrium point of the dynamics. The function \( P[i_L, v_c] \) is known as the mixed potential function.

Since \( v_\mu \) depends only on \( i_\mu \) for \( \mu > r+s \), the expression for \( P[i_L, v_c] \) can be rewritten as:

\[
P[i_L, v_c] = \sum_{\mu=r+1}^{b} \int v_\mu di_\mu + \sum_{\mu=r+1}^{b} v_\mu i_\mu \bigg|_\Gamma \tag{6.82}
\]

We can note that the integral \( \int v_\mu di_\mu \) is a well defined line integral even if \( v_\mu \) cannot be written as a single valued function of \( i_\mu \). Taken as a line integral, the path of integration is along the characteristic of the non-reactive element. This integral is called the current potential of the element in the \( \mu \)-th branch. Similarly, the line integral \( \int i_\mu dv_\mu \) is called the voltage potential of the element in the \( \mu \)-th branch. It is easily seen that

\[
\int i_\mu dv_\mu + \int v_\mu di_\mu = v_\mu i_\mu \bigg|_\Gamma \tag{6.83}
\]

The current or voltage potential has a simple interpretation if the graph of a resistor can be expressed as a single valued function of one of the variables. For example, if \( v_\mu \) is a single valued function of \( i_\mu \), then the current potential is an ordinary integral and hence the shaded area shown in figure 6.25b assuming that the initial fixed point of the path was at \( i_\mu = 0 \).

From the expression for \( P[i_L, v_c] \) in (6.80), we can write a systematic procedure for constructing the mixed potential directly from the circuit as:

1) determine the current/voltage potential for each resistor, source;
2) determine the product \( v_\mu i_\mu \) for each capacitor;
3) form the sum of these terms and express it in terms of the inductor currents and capacitor voltage.

We will now present some examples to illustrate the calculation of \( P[i_L, v_c] \).

**Example 1** Consider the circuit shown in Fig. 6.26. We have only one state variable \( i_L \), the current through the inductor. Assuming that the initial value of the current is zero, the current potential for the resistor is:

\[
\int_0^i v_\mu di_\mu = \int_0^i (-Ri_L)di_L = -\frac{1}{2} Ri_L^2 \tag{6.84}
\]

and the current potential of the battery is:

\[
\int_0^i Edi_L = Ei_L \tag{6.85}
\]

Since no capacitors are present in the circuit, we have:

\[
\int_0^i Edi_L = EP(i_L) = Ei_L - \frac{1}{2} Ri_L^2 \tag{6.86}
\]

leading to the network dynamics:
\[
v_L = \frac{d\phi_L}{dt} = -L \frac{di_L}{dt} = -\frac{\partial P(i_L)}{\partial i_L} = -(E - Ri_L) \tag{6.87}
\]

Note the negative sign in the expression for the inductor voltage which is due to the assumed polarities.

**Example 2**  Consider the tunnel diode circuit shown in Fig. 6.27a with the voltage-current characteristic for the tunnel diode as shown in Fig. 6.27b. The current potential of the resistor (tunnel-diode) is:

\[
-\int v d(f[v]) = -v_c f[v_c] + \int f[v] dv \tag{6.88}
\]

and the \( v \mu_i \) product for the capacitor is \(-(i_L - f[v_c])v_c\). Thus, the mixed potential is:

\[
P[i_L, v_c] = E i_L - v_c f[v_c] + \int_0^v f[v] dv - (i_L - f[v_c])v_c = E i_L - i_L v_c + \int_0^v f[v] dv \tag{6.89}
\]

leading to the network dynamics:

\[
\frac{d\phi_L}{dt} = -L \frac{di_L}{dt} = -\frac{\partial P[i_L, v_c]}{\partial i_L} = -(E - v_c) \tag{6.90}
\]

\[
\frac{dq_c}{dt} = -c \frac{dv_c}{dt} = -\frac{\partial P[i_L, v_c]}{\partial v_c} = -(i_L - f[v_c]) \tag{6.91}
\]

**Example 3**  Let us modify the tunnel diode circuit of example 2 by connecting a resistance \( R \) in series with the inductor as shown in Fig. 6.28a. The tunnel-diode characteristic is assumed to be given by:

\[
i_d = -i[v_d] = -v_d \left( v_d^2 - a v_d + b \right) \tag{6.91}
\]

where \( a \) and \( b \) are two positive constants. The current-voltage waveform is shown in Fig. 6.28b for some fixed values of \( a \) and \( b \). From the last two examples, the mixed potential and the dynamics of the circuit of this example can be found as:

\[
P[i_L, v_c] = E i_L - i_L v_c - \frac{1}{2} R i_L^2 + \int_0^v f[v] dv \tag{6.92}
\]

\[
\frac{d\phi_L}{dt} = -L \frac{di_L}{dt} = -\frac{\partial P[i_L, v_c]}{\partial i_L} = -(E - Ri_L - v_c) \tag{6.93}
\]

\[
\frac{dq_c}{dt} = -c \frac{dv_c}{dt} = -\frac{\partial P[i_L, v_c]}{\partial v_c} = -(i_L - f[v_c]) \tag{6.94}
\]

From the dynamics, we can note that the steady state solution is given by the intersection of the waveform

\[
E - Ri_L - v_c = E + Ri_d - v_c = 0 \tag{6.95}
\]

also shown in Fig. 6.28b and the voltage-current waveform for the tunnel diode. We can see that there are at most three intersections and hence the number of equilibria is at most 3. It can be shown that the two equilibrium points \([i_{1,1} v_{c1}]\) and \([i_{1,3} v_{c3}]\) are stable and the equilibrium point \([i_{1,2} v_{c2}]\) is unstable provided

\[
\frac{L}{R^2 c} < \frac{1}{4} \tag{6.95}
\]
In Fig. 6.29a and b, we show respectively a 3-D plot and equi-valued contour plot of the function $P[i_L,v_C]$ for some fixed values for the network element values that lead to two stable equilibria. The response of the circuit for different initial conditions are shown in Fig. 6.29c. We also show the response for another set of values for the network elements for which the response is not stable (Fig. 6.29d).

In summary, it is true from a mathematical perspective that the issue of stability of the DC response of nonlinear networks can be treated similar to the issue of stability of nonlinear networks with only initial stored energy. However, such an approach removes the connection to the physical world. In the chapter on neural networks, we will again show that the mixed potential defined here is getting minimized and not an energy or energy-like function as claimed by the neural net community.

### 6.3.3 Absolute Stability Vs. BIBO Stability: Network Interpretation

Let us consider the second-order nonlinear dynamics studied in section 4.3.3 while discussing the distinction between absolute stability and BIBO stability. The dynamics being:

$$\ddot{x} + \tanh(\dot{x}) + x = f[t]$$  \hspace{1cm} (6.96)
By equating $x(t)$ to the charge in a linear, unit-valued capacitor, we obtain a network realizing the above dynamics as shown in Fig. 6.30a. Thus, the dynamics correspond to that of a network consisting of two reactive elements (one inductor and one capacitor, both linear, time-invariant and unit-valued) and a nonlinear resistor with the voltage-current characteristics given by:

$$v_R(t) = \tanh[i_R(t)] = \begin{cases} 
\text{negative for } i_R < 0 \\
\text{zero for } i_R = 0 \\
\text{positive for } i_R > 0 
\end{cases} (6.97)$$

driven by a voltage source. Note that the above nonlinear characteristic point to a valid nonlinear, passive resistor. Therefore, the dynamics correspond to that of a passive network with the origin as the only equilibrium point and a monotonically increasing energy storage waveform. In the absence of the voltage excitation, any energy stored in the reactive elements will be consumed by the passive resistor forcing the two state variables, $q(t)$ and $\phi(t)$, to zero. Thus, the origin is the only stable equilibrium point of the nonlinear dynamics; it is asymptotically stable as the state variables moves to the equilibrium point as time tends to infinity and not just stay around the equilibrium point; it is globally, absolutely stable since each of the reactive elements have only one relaxation point and the stored energy increases monotonically as a function of the state variables.

Now consider the case when the voltage excitation is a sinusoidal one ($f[t] = A\sin[t]$) with $A$ representing the amplitude that we can vary. For $A$ very small, the resistor acts like a linear resistor leading to a linear network driven by a sinusoidal voltage source. The resulting values of the state variables will also be small. That is, the signals remain bounded when the input is bounded and of small amplitude, but persists for long time. The dynamics is therefore known as totally stable.

When the amplitude of the voltage excitation becomes larger, we find that it leads to state variables that are correspondingly larger (in magnitude) and the power drawn from (or supplied by) the ideal voltage source is proportional to square of its amplitude $A$. However, the voltage across the resistor is bounded by the value one in magnitude and hence the power consumption capacity of the resistor (the only lossy element in the circuit) becomes proportional to the voltage amplitude $A$. Thus, there is a mismatch between the power supplied (by the independent ideal source) and the power consumed (by the nonlinear resistor). The additional power gets stored in the reactive elements as stored energy. As time progresses, the power mismatch continues, the state variables increase in amplitude drawing much more power from the source and so on. The state variables therefore grow without any bound indicating that the nonlinear dynamics is not BIBO stable. Thus, mathematically speaking, a passive nonlinear network which is absolutely stable can have a dynamics that is BIBO unstable.

This example illustrates a number of important points. Nonlinear dynamics is so diverse and powerful that we can end up with different phenomena (globally, absolutely stable but not BIBO stable and so on). It also illustrates the importance of proper modeling techniques (the choice of proper elements with the proper characteristics) in representing physical elements to physical systems. After all, we moved away from linear models to nonlinear models to properly represent real-world systems, and on the other hand, choose terms that will go into that nonlinear dynamics rather arbitrarily and or assume properties that are not possible in the real-world. For example, in the above example, the damping or dissipation has been assumed to be power bounded (Fig. 6.30b). In a real-world, we can rather expect the power dissipation increase beyond that indicated by the linear model as the independent variable increases above a certain value, and enter an unstable mode that is not sustainable (burn and become a open circuit, in the case of a resistance) as the independent variable increases further as illustrated in Fig. 6.30c. Similarly, the real-world sources will be peak-power limited as shown in Fig. 6.30d and are not really capable of supplying infinite power and energy. Thus, if we are going to approximate using nonlinear and perhaps time-varying models, we should apply as much knowledge available about the process/system as possible into the modeling process. As we will discuss in later chapters, fuzzy logic and neural networks handle this issue in their own ways. Before we conclude this section, we will discuss one more example from the network literature to illustrate the importance of proper modeling in nonlinear dynamics.

**Example:** Consider the RC circuit driven by a current source shown in Fig. 6.31a. The capacitor is linear and unit valued and the resistor is assumed to be nonlinear and voltage controlled. The resistor is modeled by the current-voltage relationship given by:

$$i_R(t) = b - \frac{3}{2} [v_R(t) - a]^\frac{3}{2} (6.98)$$

for voltage in the neighborhood $\mathcal{Z}$ of a constant value 'a' (Fig. 6.31b). The dynamics of the network is given by:

$$\dot{v}_c(t) = f[v_c(t), i_0(t)]$$

$$= i_v(t) = i_i(t) - i_R(t)$$

$$= i_0(t) - b + \frac{3}{2} [v_R(t) - a]^\frac{3}{2} (6.99)$$
If we choose the current source to be a constant given by:

\[ i_0(t) = b \] (6.100)

we find that the dynamics has two solutions given by:

\[ v_{c1}(t) = a \quad ; \quad t > 0 \quad \text{and} \quad v_{c1} \in \mathbb{I} \] (6.101)

\[ v_{c2}(t) = a + t^{\frac{3}{2}} \quad ; \quad t > 0 \quad \text{and} \quad v_{c2} \in \mathbb{I} \] (6.102)

That is, it appears that a network that is passive can lead to two different responses to a single input. However, by looking at the model (small-scale) that is being used for the nonlinear resistor, we find that it has a discontinuity at \( v_R = a \).

If we choose the current source to be a constant given by:

\[ i_0(t) = b \] (6.100)

we find that the dynamics has two solutions given by:

\[ v_{c1}(t) = a \quad ; \quad t > 0 \quad \text{and} \quad v_{c1} \in \mathbb{I} \] (6.101)

\[ v_{c2}(t) = a + t^{\frac{3}{2}} \quad ; \quad t > 0 \quad \text{and} \quad v_{c2} \in \mathbb{I} \] (6.102)

That is, it appears that a network that is passive can lead to two different responses to a single input. However, by looking at the model (small-scale) that is being used for the nonlinear resistor, we find that it has a discontinuity at 'a' leading to a Jacobian for the dynamics that is discontinuous. That is:

\[ J(v_c) = \frac{\partial I}{\partial v_c} = -\frac{\partial I_R}{\partial v_R} = \frac{1}{2} (v_c(t) - a)^{-\frac{3}{2}} \quad \text{at} \quad v_c = a \] (6.103)

Thus, we may question if the problem is really in the device? (i.e., is there really a physical device with such voltage-current characteristic, and we can conclude with confidence that there will not be such a physical device) or is the
problem due to the use of improper mathematical terms (irrational function, here) for modeling physical devices? Thus, rather than writing off the important role that passivity can play in nonlinear dynamics (i.e., not getting carried away with statements such as passivity alone is not sufficient to guarantee BIBO stability or to guarantee unique solution), we should remind ourselves that passivity is one physical phenomena that has to used carefully with other physical constraints such as proper interconnection rules, use of proper models etc. to deal with nonlinear systems. In the chapter 8, we will use this philosophy to define meaningful nonlinear, time-varying elements and to study the stability of time-varying dynamics.

6.4 Summary

In this chapter, we considered the use of the various linear and nonlinear time-invariant electrical-circuit-building-blocks introduced in the last chapter in forming complex electrical networks and study the resulting dynamics. We first considered circuits made of LTI elements, the basic laws (Kirchhoff’s current and voltage law and Tellegen’s theorem) that apply when such elements are interconnected, and the restrictions that need to be observed while interconnecting such elements. We considered the I/O characteristics of such networks and discussed concepts such as impedance and admittance functions and positive real functions. We learnt that networks with lossy elements lead to stable circuits. Since the lossy elements are also linear (which imply a power consumption ability that is proportional to the square of the current or voltage), we found that such networks also have the bounded-input, bounded-output property. We then considered building complex multi-port networks in a systematic manner and studied their impedance and admittance matrices. We noticed that a state-space representation of such a network carries more information (the I/O characteristics of the individual elements & the structural information) than is indicated by stability alone. We also discussed two important techniques in LTI network theory, that of impedance scaling and frequency scaling that allows us to change the element values without affecting certain I/O characteristics of the network.

Of course, the reasons for looking at LTI electrical networks are to learn how to carry some of that knowledge to the nonlinear electrical network domain and use them effectively. In section 6.2, we considered simple networks made of nonlinear elements, the resulting dynamics and the behavior. Through these examples, we learnt how issues such as multiple equilibrium can be easily explained, or how to form complex dynamics with desired characteristics. Again, the presence of lossy elements is the key to absolute stability of the resulting dynamics. In fact, we found that the sum of the energy stored in the lossless elements (linear or nonlinear) at any given time is the true Lyapunov function for that nonlinear dynamics, and the sum of the power consumed by the lossy elements in the network correspond to the negative of the derivative of the Lyapunov function and is always non-negative. Thus, we learnt that absolute stability and passivity goes hand in hand as happens in LTI systems. We also looked at the response to initial conditions, and response to external excitation separately as they refer to two distinct physical scenarios. In the first case, we have the situation where we have finite stored energy to start with and wonder if that energy will be dissipated eventually. On the other hand, the second case implies a situation where power is being continually supplied by the external source and the question is how that in-flow of power is being handled by the network. Is it being dissipated as it comes along or gets stored? If so, how long can this continue? We found that the flexibility in defining the I/O characteristics of the lossy elements can also lead to conditions where the lossy elements are unprepared, so to speak, to absorb the power coming in leading to potentially BIBO instability. Thus, a network/dynamics that is absolutely stable may not necessarily be BIBO stable.

In the next chapter, we introduce time-varying or non-autonomous dynamics from a mathematical and analytical perspective as they are presently treated. In chapter 8, we will continue with the building block approach to define time-varying elements, and circuits made of such elements, the dynamics resulting from such networks and the stability of the dynamics. Using the element or building block approach, we show for the first time that nonlinear, time-varying dynamics does not necessarily mean that the Lyapunov function associated with such dynamics has to be time-varying (function of both the state variables and the time) as is commonly assumed.