4. Nonlinear, Time-Invariant (Autonomous) Systems

4.1 Introduction to the Chapter

In the second chapter, we noted that LTI systems are easier to characterize mathematically and the tools for the analysis and synthesis of LTI systems are well developed. However, the use of LTI systems is restricted to linear filtering. We need to consider nonlinear and time-varying systems for more complex applications. From a practical perspective, it is better to consider first nonlinear, time-invariant systems (also known as autonomous systems) and later introduce time-variation (non-autonomous systems). Thus, in this chapter, we will only discuss nonlinear autonomous systems and consider nonlinear non-autonomous systems in later chapters. In this chapter, we consider nonlinear autonomous systems from a classical mathematical perspective, the approach taken by most of the researchers in this area. However, we try to make the subject simpler and easily readable by providing a number of examples and arriving at the major results of nonlinear autonomous systems by qualitatively analyzing the resulting behavior. In chapters five and six, we will consider nonlinear systems from an electrical engineering perspective, as proper interconnection of physically realizable electrical elements with well-defined characteristics, and explain how the various elements lead to the different phenomena that make nonlinear dynamics complex and interesting.

The chapter is organized as follows: In section 4.2, we present basic concepts such as equilibrium points, stability etc. for nonlinear dynamic systems. In section 4.3, we discuss some of the well-known techniques for the analysis of nonlinear systems. In section 4.4, we consider the forced response of nonlinear systems. The separation of the response into transient and forced response, a common practice in LTI systems, is not generally followed in the nonlinear literature, and we discuss the rationale for both in that section.

4.2 Basic Concepts of Nonlinear Systems

The choice of the models to represent a physical system, the measurability of the parameters of the model, the complexity involved in their measurement, the set of inputs to be used and the responses to be measured all become very complex as we venture into the world of nonlinear (and or time-varying) systems. Hence different strategies have been developed in the study of nonlinear systems. To motivate the students properly, and to ease the entry into nonlinear systems, we first introduce the basic terminology of nonlinear systems as applied to the well-known linear systems.

4.2.1 LTI Systems Revisited Using Nonlinear Systems

Terminology

Let us begin the study of nonlinear systems by considering again the salient features and properties of commonly used special class of LTI systems while using the terminology commonly used in the nonlinear systems literature.

We noted that a LTI system could be represented in the state-space form as:

\[ \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*} \]  

(4.1a)

\[ \begin{align*}
x((n+1)T) &= \hat{A}x(nT) + \hat{B}u(nT) \\
y(nT) &= \hat{C}x(nT) + \hat{D}u(nT)
\end{align*} \]  

(4.1b)

where \( x(t) \) \{ \( x(nT) \) \} is a vector of size \( N \{ \hat{N} \} \) of the state variables, \( u(t) \) \{ \( u(nT) \) \} is a vector of size \( M \{ \hat{M} \} \) of the inputs, \( y(t) \) \{ \( y(nT) \) \} is a vector of size \( L \{ \hat{L} \} \) of the output variables, and \( A, B, C, \) and \( D \{ \hat{A}, \hat{B}, \hat{C}, \hat{D} \} \) are constant matrices of appropriate dimensions. The solution \( x(t) \) \{ \( x(nT) \) \} for \( t \geq t_0 \{ n \geq n_0 \} \) given \( x(t_0) \) \{ \( x(n_0 - 1) \) \} (initial condition representing initial stored energy in the system) and \( u(t) = 0 \{ u(n) = 0 \} \) is the transient response of the system. It can be shown as a trajectory (and hence called as state trajectory or system trajectory) in the real \( N \)-dimensional plane with the \( N \)-variables of the state vector \( x = [x_1 \ x_2 \ \ldots \ x_N]^{T} \) constituting the \( N \)-axes. For example, when \( N = 2 \), the trajectory will be on the \( x_1, x_2 \) plane as shown in Fig. 4.1. In the figure, we have shown examples of the three well-known possibilities for LTI systems:

1) A stable system;
2) A marginally stable system, and
3) An unstable system.
Using the definitions in the nonlinear systems theory literature, the plane with \( x_1, x_2 = \dot{x}_1 \) as the coordinates is known as the \textit{phase-plane}, the trajectory of the state of a two state variable system as \textit{phase-plane trajectory} and a collection of such phase plane trajectories corresponding to various initial conditions as \textit{phase portraits} of the system. In fig. 4.1 we show the phase portraits of three important cases of LTI systems. We find that in the case of stable systems all trajectories go to the origin \( (x_1 = x_2 = 0) \), go to infinity \( (x_1 \) and or \( x_2 = \infty \) ) in the case of unstable systems, and form closed contours (corresponding to periodic oscillatory behavior) in the case of marginally stable systems. It can be observed that for \( x(t_0) = 0 \) \( \{x(nT) = 0\} \):

\[
\begin{align*}
\text{Continuous system:} & \quad \dot{x}(t) = Ax(t) = 0; \quad t \geq t_0 & (4.2a) \\
\text{Discrete system:} & \quad x((n + 1)T) = \hat{A}x(nT) = 0; \quad n \geq n_0 & (4.2b)
\end{align*}
\]

whether the system is stable, marginally stable, or unstable. The value \( x = x_e \) for which \( x(t) = 0 \) for all \( t \{x(nT) = x_e \) for all \( n \} \) is called a \textit{singular point} or an \textit{equilibrium point}. In the case of stable systems, even if a disturbance occurs making \( x(t) \{x(nT) \) to move away from the equilibrium point, \( x(t) \{x(nT) \) will eventually go to the equilibrium point and stay there as and when the disturbance is removed and stay close to the equilibrium point when the disturbance amplitude is small. On the other hand, in the case of unstable systems, even a small disturbance will force \( x(t) \{x(nT) \) to infinity as \( t \to \infty \{n \to \infty \). Thus, using nonlinear systems' terminology, we can classify the equilibrium points as \textit{stable or unstable equilibrium points} depending upon the situation. Note here that we are denoting the equilibrium points and not the system itself as stable or unstable. Of course, the fact that there are only two equilibrium points \( x_e = 0 \) or \( \infty \) for LTI systems makes such definitions and classifications mute or uninteresting. In the case of nonlinear systems, we have the possibility of having more equilibrium points. Especially, the same nonlinear system may have both stable and unstable equilibrium points. Thus, the need to discuss stability in terms of the equilibrium points and not just the system.

The state vector \( x(t) \{x(nT) \) corresponds to the energy left in the system at any time, with \( x = x_e = 0 \) representing the zero energy state. Thus, \( x = x_e = 0 \) implies that the energy originally given to the system through the initial conditions or by disturbances gets dissipated as the time progresses, forcing the system state to reach the zero energy state eventually. Thus, if we draw the trajectory of the energy-left in a stable time-invariant linear system as a function of the state variables, we will get graphs as shown in figure 4.2 for one- and two-state variable systems. The energy curve takes the shape of a bowl with the minimum value of zero occurring at \( x = x_e = 0 \) and the slope of the curve being non-positive for all \( x(t) \{x(nT) \). As we will see shortly, we can have nonlinear systems with an energy curve that becomes zero and or locally minimum at more than one point (in addition to or other than \( x = x_e = 0 \) as shown in Fig. 4.2c for a first-order nonlinear system leading to \textit{multiple, stable equilibrium points}. Such curves also imply that the energy curve can have a number of local maxima as well. All these possibilities make the nonlinear system responses complicated and interesting.

When a LTI system is driven by external sources \( (u(t) \neq 0 \{u(nT) \neq 0 \) ), the resulting state and outputs \( x(t), y(t) \{x(nT), y(nT) \) are the \textit{forced response} of the system and we learnt that:
1) for a stable system, the outputs are bounded as long as the inputs are bounded (BIBO stability) and 
2) if the inputs are sinusoids, the outputs are also sinusoids of the same frequencies.

Both these situations do not carry over to nonlinear dynamical systems, as we will see shortly.

4.2.2 Nonlinear System Models: Autonomous and Non-autonomous Systems

We started the discussion of LTI systems using nonlinear systems terminology via the time-domain representation (state-space representation) though we could have used the equivalent frequency domain representation as well. However, the former can be extended to nonlinear systems in a straightforward manner where as extending the latter is possible only in very few special cases of nonlinear systems. Also, state space representation is more natural as the interaction between various elements of a system (an electrical network, for example) takes place in the time domain. Thus, using the time-domain representation, a nonlinear dynamical system can be represented in the form:

Continuous system: \( \dot{x}(t) = f[x(t), u(t), t] \) \hspace{1cm} (4.3a)

Discrete system: \( x((n + 1)T) = \hat{f}[x(nT), u(nT), n] \) \hspace{1cm} (4.3b)

where \( x(t) \) is the state vector, and \( f[.].\{ \hat{f}[.].\} \) is a nonlinear vector function of the state variables, the inputs (if any present), and the independent variable, \( t \{ n \} \), in the case of time-varying systems. For example, in the case of \( N = 2 \), we may have:

Continuous system: \( \dot{x}_1(t) = f_1[x_1(t), x_2(t), t] \)
\( \dot{x}_2(t) = f_2[x_1(t), x_2(t), t] \) \hspace{1cm} (4.4a)

Discrete system: \( x_1((n + 1)T) = \hat{f}_1[x_1(nT), x_2(nT), nT] \)
\( x_2((n + 1)T) = \hat{f}_2[x_1(nT), x_2(nT), nT] \) \hspace{1cm} (4.4b)

where \( f_1[.].\)and \( f_2[.].\{ \hat{f}_1[.].\) and \( \hat{f}_2[.].\} \) are two nonlinear functions. A well-known second-order nonlinear autonomous equation\(^1\) is the Van der Pol equation given by:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{1}{m} [c(x_1^2(t) - 1)x_2(t) + kx_1(t)]
\end{align*}
\]  \hspace{1cm} (4.5)

where \( m, k, c \) are some positive constants. Differentiating the first expression with respect to the time \( t \), the two state equations can be combined to result in a second-order nonlinear differential equation in one state-variable \( x_1 \) as:

---

\(^1\) We are tempted to use the adjective “simple” in front of the word “second-order” as would be common while describing a second-order linear system. However, we will find that even first- or second-order nonlinear systems can lead to complex and exotic behavior.

\(^2\) From here on, we will present the material in terms of continuous systems, and discuss the discrete case only if it necessary.
\[ m\ddot{x}_1(t) + c(x_1^2(t) - 1) \dot{x}_1(t) + kx_1(t) = 0 \]  
(4.6)

A number of important observations can be made from this second-order example. First, the vector function \( f[.] \) does not depend explicitly on time, the independent variable. Such systems are the equivalent of time-invariant systems in linear systems and are commonly known as \textit{autonomous systems}. For such systems, the state trajectory will be independent of the initial time. An example\(^3\) of a time-varying or \textit{non-autonomous system} is:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t)\sin[t] \\
\dot{x}_2(t) &= -\frac{1}{m}\{c(x_1^2(t) - 1)x_2(t) + kx_1(t)\}
\end{align*}
\]  
(4.7)

For such systems, the initial time has to be considered explicitly since the state-trajectory will depend upon the initial time.

The second observation that can be made from the Van der Pol model is that it is possible to combine the two state equations to arrive at a single, second-order nonlinear differential equation involving only one state-variable, \( x_1 \). Of course, this is always possible in the case of linear systems where the \( N \) state-space equations can be converted into an \( N \)-th order linear differential equation involving only and any-one state-variable. This is not always true in the case of nonlinear systems. For example, if we differentiate the second equation in (4.5) once, we will get:

\[ m\ddot{x}_2(t) + c(x_1^2(t) - 1)\dot{x}_2(t) + c(x_1(t)\dot{x}_2(t) + kx_2(t) = 0 \]  
(4.8)

where the variable \( x_1 \) cannot be eliminated to arrive at a second-order differential equation containing only the other state variable \( x_2 \). A more general second-order model with such a property is given by:

\[ \dot{x}_1(t) = f_{11}[x_1(t)] + f_{12}[x_2(t)] \]  
\[ \dot{x}_2(t) = f_{21}[x_1(t)] \]  
(4.9)

where \( f_{11}[.] , f_{12}[.] \), and \( f_{21}[.] \) are some nonlinear functions. Obviously, we cannot obtain a second-order nonlinear differential equation in one variable only since:

\[
\begin{align*}
\ddot{x}_2(t) &= \frac{df_{21}[x_1(t)]}{dt} \\
&= \frac{df_{21}[x_1(t)]}{dx_1(t)} \frac{dx_1(t)}{dt} \\
&= \frac{df_{21}[x_1(t)]}{dx_1(t)} \left(f_{11}[x_1(t)] + f_{12}[x_2(t)]\right)
\end{align*}
\]  
(4.10)

will contain \( x_i \) unless the various nonlinear functions are constrained properly. Thus, a state space description is more general and appears the most natural way to represent real-world systems, as we will see in chapters 6 & 8.

Finally, we can note that the Van der Pol equation has an equilibrium point \( x = x_e = [x_{1e} \ x_{2e}]^T = [0 \ 0]^T \) and that is the only equilibrium point for that dynamics. We will now look into the stability of this equilibrium point.

### 4.2.3 Stable, Unstable, Single, and Multiple Equilibrium Points of Autonomous Nonlinear Systems

Let us illustrate the concepts of stable and unstable equilibrium points as well as the possibility for single and multiple equilibrium points through a number of examples. Restricting ourselves to a first-order system, we can recall that a LTI system dynamics will be of the form:

\[ \dot{x}(t) = ax(t) \]  
(4.11)

where \( a \) is the only parameter that is at our disposal. Thus, we have a limited choice and from a stability point of view, we divide the range of values for the coefficient \( a \) to:

\[ 1) -\infty < a < 0 \]  
\[ 2) 0 < a < \infty \]  
(4.12a)

(4.12b)

with the first choice leading to a stable system and the second to an unstable system. The corresponding mathematical model for a first-order autonomous nonlinear system will be:

---

\(^3\) We will show in later chapters that the electrical network equivalent of Van der Pol dynamics does indeed contain a time-varying element, making it perhaps a linear time-varying dynamics. In fact, just this one example can be used to illustrate the problems arising from a pure analytical approach to nonlinear dynamical systems, and the enormous insight that can be obtained from a building-block approach used in this book, as we will do later.

\(^4\) It can be observed that this example has been derived by adding a pure time-varying component to the first state equation of the Van der Pol equation. Many models for non-autonomous systems are indeed based on this approach to prove important concepts and may not correspond to meaningful systems. Also, an autonomous system with a stand alone time-dependent function that could be considered as a forcing function will be classified as non-autonomous.
\[
\dot{x}(t) = f[x(t)]
\]  
(4.13)

Though this expression is a simple extension from the linear differential equation of (4.12), the choices and the practical implications are enormous. If we let our mathematical expertise as well as our imagination run wild and forget the connection to physical systems, we can select \( f[x(t)] \) to be a polynomial in \( x \), a rational function of \( x \), a transcendental function of \( x \), an irrational function of \( x \) and so on. The chosen function can have multiple finite valued zeros, become infinite at a number of points, have discontinuities, and may or may not have well-defined derivatives. Thus, we can expect to have multiple equilibrium points at which \( \dot{x}(t) = f[x(t)] = 0 \), complex and exotic time-domain behavior that is difficult to replicate, or behavior that depends on initial conditions etc.

Examples of some first-order nonlinear systems are:

- **System#1**: \( \dot{x}(t) = f_1[x(t)] = -x^3 \)
- **System#2**: \( \dot{x}(t) = f_2[x(t)] = x^3 \)
- **System#3**: \( \dot{x}(t) = f_3[x(t)] = -k\tanh[x] \)
- **System#4**: \( \dot{x}(t) = f_4[x(t)] = -\text{sgn}[x]x^2 \)
- **System#5**: \( \dot{x}(t) = f_5[x(t)] = -kx(1-x) \)
- **System#6**: \( \dot{x}(t) = f_6[x(t)] = -x(16x^4 - 20x^2 + 5) = -16x(x^2 - 0.3455)(x^2 - 0.9045) \)
- **System#7**: \( \dot{x}(t) = f_7[x(t)] = -\sin[x] \)

where \( k \) is assumed to be some positive constant. We can see that we indeed have almost infinite choices even for the restricted class of first-order nonlinear autonomous systems. Setting \( f_i[x(t)] = 0 \) for \( i = 1 \) to \( 7 \) and solving for \( x \), we find that systems #1, 2, 3 and 4 have one (and only one) equilibrium point at \( x_e = 0 \). System #5 has two equilibrium points at \( x_{e1} = 0 \) and \( x_{e2} = 1 \). System #6 has five equilibrium points \( (x_{e1} = 0, x_{e2} = -0.58779, x_{e3} = 0.58779, x_{e4} = -0.95106, \) and \( x_{e5} = 0.95106) \) and system #7 has infinite number of equilibrium points \( (x_{e} = m\pi) \) where \( m \) is a real integer.

Looking at some second order examples, as indicated before, the second-order Van der Pol equation has one equilibrium point given by \( x = x_e = [\theta_{e} \; \omega_{e}]^T = [0 \; 0]^T \). Another example that is worth mentioning here is that of a pendulum as shown in Fig. 4.3. Its dynamics in the state-space form is given by:

\[
\begin{align*}
\dot{\theta} &= \omega \\
\dot{\omega} &= -\frac{F}{mL^2} \omega - \frac{g}{L} \sin[\theta]
\end{align*}
\]  
(4.15)

where \( \theta \) is the angle, \( \omega \) is the angular velocity, \( m \) is the mass of the pendulum, \( L \) the length, \( F \) the friction, and \( g \) the coefficient of gravity. From the state-space equations, the equilibrium points of this system can be obtained as \( x = x_e = [\theta_{e} \; \omega_{e}]^T = [k\pi \; 0]^T, \) \( k \) real integer, suggesting infinite number of equilibrium points. Of course, in this case, all these values correspond to two real, physical locations (exactly down, and up). Thus, keeping physical concepts in mind while forming and or seeking solutions of nonlinear dynamical systems can help tremendously.

![Figure 4-3. A one degree-of-freedom pendulum leading to a second-order nonlinear dynamics.](image)
A number of important observations can be made from the examples and the responses shown in the figure. The responses of systems #1, 3, and 4 tend towards zero (the equilibrium point is stable or *stable equilibrium point*) whereas the response of system #2 tends to infinity as t increases regardless of the initial conditions (The origin is thus an *unstable equilibrium point*). For system #5, the response goes to zero (one stable equilibrium point) if \( x(0) < 1 \), stays at 1 (the other *equilibrium point which is unstable*) when \( x(0) = 1 \) and goes to infinity when \( x(0) > 1 \). The closed form solution of this nonlinear equation can be shown as:

\[
x(t) = \frac{x(0)e^{-at}}{1 - x(0) + x(0)e^{-at}}
\]

(4.16)

which for \( x(0) > 1 \) can be rewritten as:

Figure 4-4. Responses of the seven first-order systems in equation (4.14) for some initial conditions (a to g) and the response of the Vander Pol oscillator (a second-order system, with the two state variables shown in figures h1 and h2).
indicating that the denominator becomes zero (and \( x(t) \) becomes infinite) for some \( t > 0 \) and \( x(0) > 1 \).

Observing again the responses of the systems \#1 & 4 (with the origin as the stable equilibrium point), we find that the response moves closer to the origin faster when the magnitude of the state variable is greater than one and changes very slowly when the magnitude is less than one. Thus, we find that the response has not exactly reached the value zero in the amount of time (10 seconds) used for the simulation of the response.

Depending on the initial condition, the responses of system \#6 settle at \( x_1 = 0, + 0.95106 \) and \(-0.95106\) (three stable equilibrium points) and move away from \( x_1 = 0.58779 \) and \(-0.58779\) (two unstable equilibrium points). Similarly for system \#7, the response settles at \( x_2 = 2k\pi \) (k real integer), the stable equilibrium points and move away from \( x_2 = (2k+1)\pi \), (k real integer), unstable equilibrium points. Finally, the response of the Van der Pol equation neither goes to zero nor infinity but oscillates in a finite range. This oscillation is known as limit cycle oscillation. It should be observed that the amplitude of this oscillation is independent of the initial condition and can be shown to be not affected much by changes in the parameters m, c, and k since the nonlinear term \((x^2 - 1)\) plays a more significant role.

From these examples, we can note that nonlinear systems can have multiple, and non-zero valued equilibrium points with some of them stable and others unstable. Also, we can have systems such as Van der Pol equation having complex oscillatory behavior. Now we will use this knowledge to introduce the various stability concepts associated with nonlinear systems.

4.2.4 Concepts of Stability in Autonomous Nonlinear Systems

The various examples in the last section illustrate the problems involved in classifying the behavior of nonlinear systems. The responses move towards the value zero (the equilibrium point) for some systems regardless of the initial conditions, move faster for some systems & some initial conditions, and move sluggishly for some other systems & or initial conditions), move towards infinity for some other systems, move towards one of many fixed equilibrium points, move away from some of the equilibrium points, or simply becomes a bounded, complex oscillation. Thus, simple classifications as stable, or unstable, or marginally stable as we did in the case of linear time invariant systems are not sufficient. Adjectives such as local, global, and asymptotic have to be used to differentiate between the various cases. Further, since the move towards the equilibrium point(s) is (are) not uniform (like an exponential decay as in LTI systems), other definitions have to be introduced to obtain simple mathematical expressions as bounds. As we have seen that a nonlinear system can have more than one equilibrium point, and the system response tends to move towards or away from the equilibrium points, the basic concepts of stability can and has to be stated in terms of the equilibrium points as we do now.

The equilibrium state \( x = x^e \) of a nonlinear dynamics or system is said to be stable in the sense of Lyapunov stability if for any \( R_i > 0 \), there exists \( R > 0 \) such that for any given initial condition \( x(0) \) where the norm \( \|x(0) - x^e\| < R_i \), \( \|x(t) - x^e\| < R_i \) for all \( t \geq 0 \). Otherwise the equilibrium point is known as unstable.

This stability definition can be understood better using the phase-plane representation of the response of a second-order system (Fig. 4.5). It can be noted that the two state variables form the coordinates of a plane (hyper-plane in the general case), \( x(0) \) and \( x^e \) are two points in that plane, and the \( l_2 \) norm \( \|x(0) - x^e\| < R_i \) correspond to the area covered by a circle of radius \( R_i \) (sphere in the case of a 3-state variable system and hyper-sphere for \( n > 3 \)) and center \( x(0) \) and similarly for the \( l_2 \) norm \( \|x(t) - x^e\| \). If the system response is contained in the circle with center \( x^e \) and radius \( R_i \), the equilibrium point is considered as stable and unstable otherwise.

![Figure 4-5. Stability concepts based on a second-order system.](image)

\[ x(t) = \frac{x(0)e^{-at}}{x(0)e^{-at} - (x(0) - 1)} \]  

(4.17)
Using this basic definition and the examples we have seen we can arrive at various sub-categories. We can note that the equilibrium points of systems # 1, 3, and 4 are stable and remain so regardless of how small or large the specified values of \( R_i, R_f \) are. In fact, for these systems, \( x(t) \to x_e = 0 \) as \( t \to \infty \) for any initial state and not just lie around \( x_e \). Such equilibrium points and the corresponding systems can be classified as \textit{globally, asymptotically stable} (asymptotic stability implies actual convergence to \( x_e \) as \( t \) goes to infinity rather than just staying close to it; global indicates that the asymptotic stability holds for any arbitrary initial state \( x(0) \) in the real N-dimensional space). On the other hand, the equilibrium point of system # 2 is \textit{globally unstable}. The equilibrium point \( x_e = 0 \) of system # 5 for which \( x(t) \to x_e = 0 \) as \( t \to \infty \) when \(-\infty < x(0) < 1\) can be classified as \textit{locally, asymptotically stable}. The values \(-\infty < x(0) < 1\) become the points (domain, in the multi state-variable case) of attraction for the local asymptotic stable equilibrium point. The equilibrium point \( x_e = 1 \) corresponds to an \textit{unstable equilibrium point (repelling equilibrium point)} since all trajectories (except when \( x(0) = 1 \)) move away from it. Similarly for system # 6, the equilibrium points \( x_e = 0, +0.95106, -0.95106 \) are locally asymptotically stable where as \( x_e = +0.58779, -0.58779 \) correspond to unstable equilibrium points. The region of attraction of the different equilibrium points of systems # 1 to 6 is shown in fig. 4.6. Finally, systems such as the Van der Pol equation that exhibit limit cycle behavior can be classified as unstable as we cannot expect the response to remain arbitrarily close to the equilibrium point or marginally stable.

The presence of multiple, stable equilibrium points may or may not be a problem depending upon the situation. For example, if such a model represents a satellite’s dynamics, the presence of multiple, stable equilibrium points implies that the satellite may move from one (desired) equilibrium point to another (undesirable) equilibrium point when disturbances of sufficient magnitude occur. Naturally in such problems, multiple equilibrium points are highly undesirable and must be eliminated (from the open-loop system) by adding suitable feedback control.

\textbf{4.2.4.1 Exponential Stability}

The above definitions do not take into consideration how fast or how slow the response will be in reaching the equilibrium point, assuming that the equilibrium point is a stable one. Recall that in a stable LTI system, the transient response is given by a weighted-sum of complex exponentials:

\[
h_{LTI}(t) = \sum_{i=1}^{N} k_i e^{s_i} u_j(t) = \sum_{i=1}^{N} k_i e^{(-\alpha_i + j\omega_j)} u_j(t)
\]  

(4.18)
where \( s_{pi} \) are the roots of the deterministic polynomial \( \det [sI - A] \) and for absolutely stable systems, the real parts \( s_{pi} \) are negative forcing the transient response to go to zero as \( t \) goes to infinity. Further, the rate of decay or how fast the response reaches the equilibrium point is exponential, and is determined by two values, \( k_{\text{max}} \) and \( \sigma_{\text{min}} \), where:

\[
  k_{\text{max}} = \text{Max}[k_i], \quad \sigma_{\text{min}} = \text{Min}[\sigma_i]
\]

(4.19a)

and

\[
  \|h_{\text{LTI}}(t)\| \leq k_{\text{max}} e^{-\sigma_{\text{min}} t} \quad \text{for all } t > 0
\]

(4.19b)

Thus, the absolute stability and exponential decay to the equilibrium point go hand in hand in the case of LTI systems. Further, we can use these two values to describe the rate of convergence to the equilibrium point regardless of how complex the system is.

Naturally, the solution of nonlinear differential equations (to initial conditions) will take a more complex form if a closed-form solution can indeed be found, and the presence of nonlinearities affects the rate of convergence with in the same system as time progresses. An example is a simple first-order nonlinear dynamics given by:

\[
  \dot{x} + x^2 = 0
\]

(4.20)

that has a stable equilibrium point at \( x_e = 0 \). It is easy to see that this equilibrium point is globally asymptotically stable. However, we find that when \( |x(t)| > 1 \), \( |\dot{x}(t)| \) is so large that the dynamics forces the response to move towards the equilibrium point rapidly. But when \( |x(t)| < 1 \), \( |\dot{x}(t)| \) becomes too small and the dynamics makes the magnitude of \( x(t) \) to decrease rather very slowly. Thus, it will take a very large time for the response to reach the equilibrium point and in fact, an enormous amount of time is needed for the difference between the response and the equilibrium point to become insignificant. Thus, asymptotic stability alone is not sufficient for real-world applications. We need to add additional constraints to describe the limiting behavior as in the case of LTI systems. For example, we could require the response envelope to be bounded by an exponential function leading to the concept of exponential stability. We can characterize a nonlinear system as exponentially stable if there exists two strictly positive numbers \( \alpha, \beta \) such that:

\[
  \|x(t) - x_e\| \leq \alpha \|x(0)\| e^{-\beta t} \quad \text{for all } t > 0
\]

(4.21)

Of course, modifiers such as local and global need to be and can be added depending on the regions of \( x(0) \) for which this condition is applicable. It is needless to say that we can’t find the two strictly positive numbers \( \alpha, \beta \) satisfying equation (4.21) for the dynamics (4.20), and hence the dynamics is not exponentially stable.

Note that the concept of exponential stability is more involved (indicates that the system state not only reaches the equilibrium point, but indicates the rate of convergence to the equilibrium point) than asymptotic stability (just indicates that the system reaches the equilibrium point). Thus, exponential stability implies asymptotic stability whereas the reverse is not true.

Examples that illustrate the differences between asymptotic stability and exponential stability are mostly based on first-order models whose solutions can be written in a closed-form. For example, the solution for first-order nonlinear differential equations of the form:

\[
  \dot{x}(t) + x(t)f[x(t)] = 0
\]

(4.22)

with \( f[x] \) some function of \( x \), is given by:

\[
  x(t) = x(0)e^{-\int_{0}^{t} f[x(t)] \, dt}
\]

(4.23)

we can form a number of examples with or without the exponential stability property by selecting suitable functions for \( f[x] \). From the above expression, we can note that the integral of the function \( f[x] \) has to be positive and bounded for any value of \( t \). For instance, if

\[
  f[x(t)] = 1 + \sin^2[x(t)]
\]

(4.24)

the solution, \( x(t) \), is given by:

\[
  x(t) = x(0)e^{-\int_{0}^{t} \sin^2[x(t)] \, dt} = x(0)e^{-\int_{0}^{t} g[t(t)] \, dt} \quad \text{where } 1 \leq g[t] \leq 2
\]

(4.25)

or

\[
  |x(t)| \leq |x(0)| e^{-t} \quad \text{for all } t
\]

(4.26)

That is, \( \alpha = \beta = 1 \) for this system. On the other hand, if we choose \( f[x] = x \) and \( t \geq 0 \), and \( x(0) > 1 \), the solution is given by:

\[
  x(t) = x(0)e^{-t}
\]
\[ x(t) = \frac{x(0)}{1 + t} \] (4.27)

which decays slower than any exponential function \( e^{-\beta t} \) with \( \beta > 0 \). Hence this system is not exponentially stable.

It is obviously not possible to obtain analytically the value of \( \alpha, \beta \) for higher-order systems. In such cases, one has to resort to estimation by computer simulation.

We summarize the results of this section on stability of nonlinear systems as a flow-chart in Fig. 4.7.

4.3 Autonomous Nonlinear System Analysis Tools

From the examples and the accompanying discussions, the readers should have realized that it is impossible to obtain closed-form solutions of nonlinear dynamical systems except in very simple cases. Thus, each (class of) system has to be handled differently and various techniques have been proposed in the literature. The graphical approach, one of the well-known analysis tools, has been introduced to characterize mainly second-order systems. Analytical methods based on Lyapunov’s linearization approach and Lyapunov’s direct method are applicable to a larger class of problems, though they do have a number of limitations. The emergence of very high-speed digital computers has also made possible characterization through simulations. In this section, we will present the basics of these nonlinear systems analysis tools, their use, and the limitations.

Figure 4-7. A summary of various types of responses leading to different stability definitions for nonlinear dynamic systems.
4.3.1 Graphical Approach for the Analysis of Nonlinear Systems

We have already seen some examples of the graphical approach. The basic idea here is to generate the state trajectories corresponding to various initial conditions, and analyze the resulting trajectories for important characteristics of the systems.

There are two problems associated with this approach. As the number of state variables increases, the computational complexity associated with the calculation of the system trajectories also increases. However, this is no longer a major problem for reasonable order systems given the advances in the computing area and their easy availability. The second problem is due to the geometrical complexity. Even with all the talk about research in “visualization software and techniques”, and the advances in these areas, our ability to represent the state trajectories in a graphical form is limited to two or three state systems. It is quite possible that we may never overcome this problem. Thus, graphical approach for the analysis of nonlinear systems is (and will be) limited to second- (and perhaps, third-) order systems. Still, it is worthwhile to understand this approach as many practical systems can be represented by second- and third-order models. Further, a graphical approach allows analysis of systems with small or smooth nonlinearities as well as hard nonlinearities. We now present a number of examples to bring out the various, important properties of nonlinear systems.

Example: 1. First-order systems

The phase portrait of a first-order system is simply obtained by plotting \( \dot{x}(t) \) as a function of \( x(t) \), the state variable for various values of \( t \). We have already looked at such plots for a number of such systems (Fig. 4.6). Considering again, system \# 6 given by:

\[
\dot{x}(t) = -16x(t)(x(t) - 0.58777)(x(t) + 0.58777)(x(t) - 0.95106)(x(t) + 0.95106)
\]  
(4.28)

and its phase portrait in Fig. 4.6f, we can see from the arrows in the graph, the values of \( x = 0.0, \pm 0.95106 \) are the stable equilibrium points and \( x = \pm 0.58777 \) are the unstable equilibrium points. The corresponding domains of attraction for the three stable equilibrium points can also be seen in the figure.  

In this plot as well the other plots in figure 4.6, the waveforms may appear to be not continuous near the equilibrium points. These waveforms were obtained by simulating the systems’ responses (and not just plotting the nonlinear functions \( f_i[x, i] \) for a fixed amount of time, and the responses do not reach the equilibrium point, indicating that the equilibrium points may not be exponentially stable).

The curious reader may notice that this system has been formed by setting \( \dot{x}(t) \) equal to the negative of a fifth-order Tchebyshev polynomial in \( x(t) \) to produce the multiple equilibrium points. Such an approach, which is common in the classical nonlinear systems theory area, does not take into consideration if physical systems corresponding to such models are really possible. In the next chapter, we will consider the physical implications to arrive at proper, useful models.

Example: 2. Second-Order LTI Systems

Let us consider the case of second-order linear, time-invariant systems, and their phase portraits (seen earlier in Fig. 4.1), before we look at the nonlinear case. In the case of absolutely stable LTI systems, we know that the transient responses become zero in an exponential manner leading to a phase portrait as shown in Fig. 4.1a. On the other hand, for an unstable system the transient responses move away from the origin, and move towards infinity for any initial condition leading to the phase portrait in Fig. 4.1c. Finally, in the case of a marginally stable LTI system or an oscillator, we have the dynamics given by:

\[
\ddot{x}(t) + \omega_0^2x(t) = 0
\]  
(4.29)

where \( \omega_0 \) is the frequency of oscillation. For a given initial condition \( x(0), \dot{x}(0) \), integration of this equation with respect to \( x \) yields:

\[
\int\{\ddot{x}(t) + \omega_0^2x(t)\}dx = \int\{\frac{dx(t)}{dt} + \omega_0^2x(t)\}dx = \int \dot{x}d\dot{x} + \omega_0^2x(t)dx = \frac{\dot{x}^2(t)}{2} + \omega_0^2\frac{x^2(t)}{2} + c
\]  
(4.30)

where \( c \) is a constant. Since the above equation is valid for all \( t \), we can use the given initial conditions to arrive at the following expression:

\[
\omega_0^2\dot{x}^2(t) + \dot{x}^2(t) = -2c = \omega_0^2\dot{x}^2(0) + \dot{x}^2(0) \quad \text{forall} \; t
\]  
(4.31)

That is, the two state variables are constrained to lie on a circle (or ellipse) leading to a periodic response. The phase portrait of this system shown in Fig. 4.1b consists of circles of varying radii as expected indicating the dependence of the response on the initial conditions.

Example: 3. Van der Pol Dynamics; A second-Order Nonlinear System

We have seen the Van der Pol dynamics earlier and noted that the dynamics has a periodic response known as limit cycle oscillation. The phase portrait of this dynamics is shown in Fig. 4.8. Note that unlike the case of LTI oscillatory
systems the response for all initial conditions ends in a single closed contour as time progresses. We will provide an interpretation from electrical networks' perspective for this difference in later chapters.

Example: 4. Another second-Order Nonlinear System
Consider a system given by:

\[ \ddot{x}(t) + c \dot{x}(t) + k_1 x(t) + k_2 x^3(t) = 0 \text{ with } c = 0.5; k_1 = 1; \text{ & } k_2 = 0.1 \]  

(4.32)

This system has only one equilibrium point, \((0 \ 0)\), the origin, which is globally stable. The phase portrait of this system appears as shown in Fig. 4.9 where all the trajectories move towards the origin.

Example: 5. Another Second-Order Nonlinear System
Consider the nonlinear system:

\[ \ddot{x}(t) + c \dot{x}(t) + k_1 x(t) + k_2 x^2(t) = 0 \text{ with } c = 0.5; k_1 = 4; \text{ & } k_2 = 1 \]  

(4.33)

The difference between this system and the previous one is that the (only) nonlinearity has been changed from \(x^3(t)\) to \(x^2(t)\) along with the coefficient associated with it. We can observe that the system has two equilibrium points, the origin \((0 \ 0)\), and \([-4 \ 0]\). The phase portrait is shown in Fig. 4.10. It is obvious from the figure that the origin is a stable equilibrium point whereas the other equilibrium point \([-4, 0]\) is not.

Figure 4-8. a) & b) The response of the Vander Pol dynamics for various initial conditions. c) Phase plane plot of the response.

Figure 4-9. Phase portrait of a second-order nonlinear system with the origin as the only equilibrium point and a globally stable one.
Consider the nonlinear system:

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
- \begin{bmatrix}
x_1(t) (x_1^2(t) - 1) \\
-x_1(t) (x_2(t) - 1)(x_2(t) - 2)
\end{bmatrix}
\]  
(4.34)

This system has nine equilibria given by the combinations of \(x_{1e} = -1, 0, 1\) and \(x_{2e} = 0, 1, 2\). The phase portrait of this dynamics is shown in Fig. 4.11. We can observe from the figure that all the trajectories go to one of the four equilibria, \([-1, 0], [-1, 2], [1, 0]\) and \([1, 2]\), indicating that they are the stable ones and the other five equilibria, including the origin, are not stable.

Example: 6. Another Second-Order Nonlinear System

Example: 7. Second-Order Nonlinear Systems with Limit Cycles

Earlier, we came across the Van der Pol equation and its response, and noted that it has a periodic response known as a limit cycle. In this example, we will look at three more nonlinear system models\(^7\) similar to the Van der Pol equation and learn about different kinds of limit cycles that are possible in nonlinear systems. We start with a dynamics as given below:

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
x_2(t) - x_1(t)(x_1^2(t) + x_2^2(t) - 1) \\
-x_1(t) - x_2(t)(x_1^2(t) + x_2^2(t) - 1)
\end{bmatrix}
\]  
(4.35a)

or

\[\text{We will explain in chapter # 8, how such and other models can be arrived at using network considerations.}\]
Note that unlike the linear case, defining self-feedback and cross-feedback terms is not easy and is subject to different interpretations.

\[
\begin{bmatrix}
    \dot{x}_1(t) \\
    \dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 \\
    -1 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t)
\end{bmatrix} -
\begin{bmatrix}
    x_1(t)(x_1^2(t) + x_2^2(t) - 1) \\
    x_2(t)(x_1^2(t) + x_2^2(t) - 1)
\end{bmatrix} \tag{4.35b}
\]

In the above equation, in addition to writing the dynamics in the traditional form (4.35a), we have also used a different representation (4.35b). In this representation, the Cross-feedback terms and self-feedback terms have been separated \(^8\) with the corresponding flow-graph representation as given in Fig. 4.12. In this particular example, the vector corresponding to the cross-feedback terms are linear, and in the absence of the self-feedback terms makes the system a LTI lossless system with an oscillatory response given by:

\[
x_1^2(t) + x_2^2(t) = 1 \tag{4.36a}
\]

The self-feedback terms (written in the form of negative feedback) involves multiplication by a nonlinear term given by:

Using similar arguments, we find that the nonlinear system (4.37) becomes stable when \(x_1^2(t) + x_2^2(t) < 1\) (and hence the system trajectories reach origin, the equilibrium point), unstable when \(x_1^2(t) + x_2^2(t) > 1\) (the response and the trajectories diverge) and stays on the unit circle (the limit cycle behavior) when \(x_1^2(t) + x_2^2(t) = 1\) (Fig. 4.13b). However, any small disturbance to the system would make the system state move away from the limit cycle response\(^9\). Thus, such limit cycles can be termed as unstable limit cycles or repelling limit cycles.

\[f[x_1(t), x_2(t)] = 1 - x_1^2(t) - x_2^2(t) \tag{4.36}\]

which when set equal to zero takes the same form as the response without the self-feedback terms. This nonlinear term becomes negative for \(x_1^2(t) + x_2^2(t) < 1\), making the self-feedback positive, the system unstable, and hence the response to move away from the equilibrium point, the origin. But as the response increases to a level where \(x_1^2(t) + x_2^2(t) > 1\), the self-feedback becomes negative, the system stable, and hence the response starts contracting. When the limiting case \((x_1^2(t) + x_2^2(t) = 1)\) is reached, the nonlinear terms and hence the self-feedback disappear, the response becomes oscillatory as given by equation (4.36a) and remain so for ever. The phase portrait of this system is shown in Fig. 4.13a. Since any initial condition will force the response to the limit cycle behavior given by equation (4.36a), such a limit cycle is known as stable limit cycle or an attracting limit cycle.

From the above discussions, it is perhaps obvious to the readers how one can skillfully hand-code nonlinear systems to lead to systems with limit cycle behavior. In fact, we can carry this process further by making some simple/strategic changes to this example. For instance, we can:

1) simply change the sign of the self-feedback terms, and
2) take the absolute value of the nonlinear terms

to obtain two different nonlinear systems as given below:

\[
\begin{bmatrix}
    \dot{x}_1(t) \\
    \dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 \\
    -1 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t)
\end{bmatrix} -
\begin{bmatrix}
    |x_1(t)(1-x_1^2(t)-x_2^2(t))| \\
    |x_2(t)(1-x_1^2(t)-x_2^2(t))|
\end{bmatrix} \tag{4.37}
\]

\[
\begin{bmatrix}
    \dot{x}_1(t) \\
    \dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 \\
    -1 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t)
\end{bmatrix} -
\begin{bmatrix}
    x_1(t)|1-x_1^2(t)-x_2^2(t)| \\
    x_2(t)|1-x_1^2(t)-x_2^2(t)|
\end{bmatrix} \tag{4.38}
\]

\(^8\) Note that unlike the linear case, defining self-feedback and cross-feedback terms is not easy and is subject to different interpretations.

\(^9\) In our simulation, we found it necessary to increase the precision used to represent the variables so as to keep the trajectory on the limit cycle when starting with an initial point on the limit cycle.
Finally, equation (4.38) represents a system that is stable when $x_1(t) + x_2(t) < 1$ or $x_1(t) + x_2(t) > 1$, and a lossless LTI system when $x_1(t) + x_2(t) = 1$. Thus, any value of $x_1(0), x_2(0)$ where $x_1(0) + x_2(0) < 1$ would push the system towards the origin where as values $x_1(0), x_2(0)$ where $x_1(0) + x_2(0) > 1$ would push the system towards the limit cycle, and any value of $x_1(0), x_2(0)$ where $x_1(0) + x_2(0) = 1$ would keep the state of the system on the limit cycle itself (Fig. 4.13c). Of course, whether the system comes back to the limit cycle behavior after a disturbance will depend on whether the resulting system state is outside or inside of this limit cycle. Thus, such limit cycles are known as semi-stable limit cycles.

In summary, the three models presented here, in addition to illustrating the use of the graphical approach, also demonstrate that the limit cycle behavior of nonlinear systems can be more complex than is possible from marginally stable LTI systems.

4.3.2 Lyapunov’s Linearization Method

This method, due to Lyapunov, is perhaps due to success in dealing with linear, time-invariant systems. The key concept is that if a nonlinear dynamics can be linearized around an equilibrium point (say $x_e = 0$), LTI system analysis tools can be used to study the properties of the resulting LTI system. Hopefully, conclusions can be drawn about the nonlinear system from those properties.

The basic tool for linearization is the mathematician’s venerable and omnipotent Taylor series expansion (TSE) of the nonlinear function $f(x)$ around the equilibrium point $x_e$. Recall that the TSE for a function $f(x)$ of a single variable $x$ around $x = x_0$ is given by:

$$f(x = x_0 + \tilde{x}) = f[x_0] + \frac{\tilde{x}}{1!} f'[x_0] + \frac{(\tilde{x})^2}{2!} f''[x_0] + \cdots$$  \hspace{1cm} (4.39a)

where $\tilde{x}$ is assumed to be small. Thus, the autonomous nonlinear system dynamics can be written using TSE as:
\[ \dot{x}|_{x = x_\epsilon} = f[x = x_\epsilon + \tilde{x}] = f[x_\epsilon] + \left[ \frac{\partial f}{\partial x} \right]_{x = x_\epsilon} \tilde{x} + \text{higher order terms} \]

\[ \left[ \frac{\partial f}{\partial x} \right]_{x = x_\epsilon} = \begin{bmatrix} A \
\end{bmatrix} [\tilde{x}] \]

where we make the standard assumption that \( f[x] \) is continuously differentiable in \( X \) so that we can express \( f[x] \) in a TSE form, the elements of the vector \( \tilde{x} \) are small in amplitude, and use the fact that \( f[x_\epsilon] = 0 \) and arrive at the approximate linear model by dropping the higher-order terms in \( \tilde{x} \) of the TSE. The constant matrix \( A \) given by:

\[ A = \left[ a_{ij} \right]_{i,j=1}^{1 \times N} = \left[ \frac{\partial f}{\partial x} \right]_{x = x_\epsilon} \]

is known as the Jacobian matrix of \( f[x] \) with respect to \( X \) at \( x = x_\epsilon \).

The linearized system \( \dot{x} = \dot{\tilde{x}} = [A][\tilde{x}] \) can be examined for stability by calculating the eigenvalues of the \( A \) matrix (the zeros of the characteristic polynomial, \( \det(AI - A) \)). Obviously, we will have three possibilities:

1. Linearized system strictly stable (all eigenvalues in the LHS of the complex plane)
2. Linearized system is unstable (one or more eigenvalues on the RHS or multiple eigenvalues on the imaginary axis of the complex plane)
3. Linearized system is marginally stable (some imaginary valued, simple eigenvalues with the rest on the LHS of the complex plane).

The question is how to use this knowledge in interpreting the stability of the original nonlinear system.

If the linearized system is strictly stable, the equilibrium point \( x_\epsilon \) is locally, asymptotically stable because the effects of the neglected higher order terms will be negligible near \( x_\epsilon \). The word “locally” is very important since the TSE is applicable only in the vicinity of \( x_\epsilon \). Similarly, if the linearized system is unstable, the equilibrium point \( x_\epsilon \) is unstable. Finally, if the linearized system is marginally stable, nothing much can be said about the stability of the equilibrium point since the neglected higher order terms may have greater influence. From this last statement, the readers should realize that there is a difference between the strict mathematical definitions of stability and instability and marginal stability and qualitative interpretation based on their exact values.

As a concrete example, a linearized system with two eigenvalues given by \(-0.001+ j100\) and \(-0.001-j100\) with the rest of the eigenvalues well in the LHS of the complex plane may be considered to be strictly stable or marginally stable (or even unstable) depending upon the amount of confidence we have on the computer and the software used to calculate the eigenvalues. Here, we need to add the possible effects of approximation based on the TSE.

**Example** Let us consider the second-order dynamics seen in example 6 and reproduced below:

\[ \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t)(x_2(t)-1)(x_2(t)-2) \\ x_1(t)(x_2(t)-1) - x_2(t)(x_2(t)-1)(x_2(t)-2) \end{bmatrix} \]

The matrix \( \frac{\partial f}{\partial x} \) and the Jacobian matrix \( A \) at two equilibrium points, \( x_{1e} = 0, x_{2e} = 0 \) and \( x_{1e} = 1, x_{2e} = 0 \), are:

\[ \left[ \frac{\partial f}{\partial x} \right]_{x = x_{1e}, x_{2e} = 0} = \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix} \]

The Jacobian matrix at \( x_{1e} = 0, x_{2e} = 0 \) points to an unstable LTI system whereas the one at \( x_{1e} = 1, x_{2e} = 0 \) corresponds to a stable LTI system. Therefore, \( x_{1e} = 0, x_{2e} = 0 \) is unstable whereas \( x_{1e} = 1, x_{2e} = 0 \) is locally stable.

### 4.3.3 Lyapunov’s Direct Method

It is obvious that the Lyapunov linearization method does not lead to reliable conclusions about a given system in all cases. A better method is needed. Lyapunov’s Direct Method (LDM) fills that need. It is perhaps based on the observation that in a physical world, some energy (a scalar function of the state variables) is associated with all physical systems. Further, it gets continuously dissipated in stable systems when no other external energy sources are applied forcing the response to settle at an equilibrium point (zero energy state). In the case of mechanical systems, the energy will be a sum of the kinetic energy and the potential energy stored in moving mass and springs and dissipation resulting from dampers. In the case of electrical systems, the total energy will be the sum...
of the electrical and magnetic energy stored in the lossless (linear or nonlinear) reactive elements and dissipation of energy will result from lossy elements such as (linear or nonlinear) resistors. Thus, if the physical system (whose nonlinear dynamics is being studied) is known, its energy storage, generation, and dissipation properties can be studied to understand the given nonlinear dynamics\(^\text{10}\). If only the mathematical description (in the form of nonlinear state equations) is given, then the stability of the nonlinear dynamics can be inferred by forming a scalar function \(V(x)\) of the state variables that has the property of an energy function (\(V(x) > 0\) when \(x \neq x_e\) and \(V(x) = 0\) when \(x = x_e\)) and examining the derivative of that function) evaluated along the system trajectory. That is:

\[
\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} - \frac{\partial V}{\partial t} \bigg|_{x = x(t)} = \frac{\partial V}{\partial x} f(x) \tag{4.41}
\]

will correspond to the rate of energy addition or dissipation by the system corresponding to that nonlinear dynamics. Thus, if \(V(x) < 0\) for \(x \neq x_e\) and zero for \(x = x_e\)\(^\text{11}\), it implies that the power is continuously dissipated, and hence the nonlinear dynamics is a stable one. A function \(V(x)\) with such properties is known as a **Lyapunov function**.

**The original Nonlinear System of Lyapunov**

The original nonlinear system that seems to have led to the Lyapunov’s Direct method, a mathematical approach for determining stability, is a mechanical system consisting of a mass, nonlinear spring, and a nonlinear damper as shown in Fig. 4.14a. The mass is restricted to travel in only one axis (single or one degree of freedom), the x axis, with \(x = 0\) corresponding to the at-rest position for the spring. The model for the nonlinear spring has been chosen to be:

\[
f_s(x(t)) = k_0x(t) + k_1x^3(t) \tag{4.42a}
\]

where \(f_s(x)\) is the force on the spring at any distance \(x\) and the two parameters \(k_0, k_1\) are assumed to be positive. Similarly, the model for the damper has been chosen to be:

\[
f_d(\dot{x}(t)) = b\dot{x}(t)\dot{x}(t) \tag{4.42b}
\]

with \(b\) positive and \(f_d(\dot{x})\) the force needed to move the nonlinear damper. The waveforms \(f_s(x)\) and \(f_d(\dot{x})\) are also shown in Fig. 4.14 assuming some fixed values for the coefficients. It can be observed that the waveforms

1) are confined to the first- and third-quadrants,
2) pass through the origin, and
3) are anti-metric, i.e.

\[
f_s(-x(t)) = -f_s(x(t))
\]

and

4) are monotonically increasing.

The energy (potential energy) contained in the spring for any value \(x\) and the energy spent (or work done) in moving the damper for any value \(\dot{x}\) can be obtained by integrating equations 4.42a and 4.42b with respect to the corresponding independent variables:

\[
E_s = \frac{1}{2} k_0x^2 + \frac{1}{4} k_1x^4 \Rightarrow \begin{cases} > 0 & \text{for } x > 0 \\ = 0 & \text{for } x = 0 \end{cases}
\]

\[
W_d = \int_0^x f_d(\dot{x}) \, d\dot{x} \tag{4.44b}
\]

\[
= \begin{cases} > 0 & \text{for } |\dot{x}| > 0 \\ = 0 & \text{for } |\dot{x}| = 0 \end{cases}
\]

\(^\text{10}\) In fact, the methods proposed in this book are based on such a concept and take it one step further— to the element or building block level.

\(^\text{11}\) Such a function is known as a positive definite function. If there is only one stable equilibrium point \(x_e = 0\), then the energy function becomes globally positive definite. If there are more than one stable equilibrium point, then the function will be called locally positive definite or globally positive semi-definite (that is, \(V(x)\) becomes zero for \(x = x_{ei} \neq 0\) for \(i = 1, 2, \ldots\)).

\(^\text{12}\) Such a function is known as negative definite.
and are shown in Fig. 4.14 b & c respectively. We can thus see the role of the four constraints on the spring and the damper characteristics. Confining the waveforms to first- and third-quadrants ensures that the energy stored or energy spent is positive, as it should be for physical, devices. Forcing them to go through the origin ensures that the point \( x = 0 \) (or \( \dot{x} = 0 \)) is the relaxation point for the spring (value at which no work is done, in the case of the damper). The third characteristic makes the energy the same regardless of the sign of the independent variable, a property that may or may not hold in practice, but seems to be an obvious choice as a first order approximation. The fourth property leads to a bowl shaped energy function with only one minimum (In the next chapter, we will define physical nonlinear devices for which this property may not necessarily be true, leading to more complex dynamics).

If the mass is pulled away from the natural length of the spring \( (x \neq 0) \) and released, we inject initial energy into the system that is the sum of the kinetic energy in the mass given by:

\[
E_m[\dot{x}] = \frac{1}{2} m \dot{x}^2 \tag{4.45}
\]

and the potential energy in the spring as seen before. Thus, the energy at any time instant (corresponding to the state \( x^i = [x, \dot{x}] \)) is given by:

\[
E[x, \dot{x}] = E_m[\dot{x}] + E_s[x] = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k_0 x^2 + \frac{1}{4} k_1 x^4 \tag{4.46}
\]

a scalar variable of the state (and hence the time) and zero energy correspond to the point \( x^i = [x, \dot{x}] = [0, 0] \). The dynamics of the mass-spring-damper system can be written as:

\[
m \ddot{x} + b \dot{x} + k_0 x(t) + k_1 x^3(t) = 0 \tag{4.47}
\]

which has the origin as the equilibrium point. Thus asymptotic stability (defined from the state behavior perspective) which implies that all state trajectories move toward the equilibrium point also corresponds to the requirement that the energy in the system gets slowly dissipated and eventually become zero. That is, the asymptotic stability requirement can be stated in terms of the energy left in the system as:

\[
\frac{dE}{dt} \leq 0 \tag{4.48}
\]

From the energy equation (4.46) and the system dynamics in equation (4.47), the time-derivative of the energy function can be calculated as:

\[
\frac{dE}{dt} = m \dot{x} \frac{d\dot{x}}{dt} + (k_0 x + k_1 x^3) \frac{dx}{dt} = m \ddot{x} + (k_0 \dot{x} + k_1 \dot{x}^3) \dot{x} = -b |\dot{x}| \dot{x}^2 \leq 0 \tag{4.49}
\]
Note that we have substituted the system dynamics in the equation for the time derivative of the energy function. This is known as evaluation of the time-derivative along the system trajectory. Equation (4.49) does indicate that the initial energy given to the system will get spent (or at least no new energy will be added) due to the motion of the system. Thus, we can expect the system state to move towards the equilibrium point and not away from it.

The curious reader would have noticed that the derivative is not negative definite, but negative semi-definite implying that the state may get stuck at a point other than the origin. We will present few more examples first and later indicate how this problem is resolved.

Examples: Consider the three second-order dynamics given below:

\[ \ddot{x}(t) + a_1 \tan^{-1}(\dot{x}(t)) + b_1 x(t) + c_1 x^3(t) = 0; \quad a_1, b_1, c_1 > 0 \]  \hspace{1cm} (4.50a)

\[ \ddot{x}(t) + a_2 \dot{x}(t) + b_2 \sin(x(t)) = 0; \quad a_2, b_2 > 0 \]  \hspace{1cm} (4.50b)

\[ \dot{x}_3 \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f_1[x_1(t)] = f_1[x_3] \]  \hspace{1cm} (4.50c)

where \( f_1[\cdot], f_2[\cdot], \) and \( f_3[\cdot] \) are functions (linear or nonlinear) confined to first- and third-quadrants and pass through the origin. The first two equations can be written in the form of state equations as:

\[ \dot{x}_1 \equiv \begin{bmatrix} \dot{x}(t) \\ \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ -b_1 x(t) - c_1 x^3(t) - a_1 \tan^{-1}[x_1(t)] \end{bmatrix} = f_1[x_1] \]  \hspace{1cm} (4.51a)

\[ \dot{x}_2 \equiv \begin{bmatrix} \dot{x}(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ -b_2 \sin[x(t)] - a_2 x_1 \end{bmatrix} = f_2[x_2] \]  \hspace{1cm} (4.51b)

where \( f_1[x_1], f_2[x_2], f_3[x_3] \) are three nonlinear vector functions of the states \( x_1, x_2, \) and \( x_3 \) respectively. The three systems have the origin as an equilibrium point. The second system also has infinite number of non-zero states as equilibrium points. The Lyapunov function candidates\(^{13} \) respectively are:

\[ V_1[x_1] = \frac{1}{2} x_1^2 + \frac{b_1}{2} x^2 + \frac{c_1}{4} x^4 \]  \hspace{1cm} (4.52a)

\[ V_2[x_2] = \frac{1}{2} x_1^2 + b_2(1 - \cos(x)) \]  \hspace{1cm} (4.52b)

\[ V_3[x_3] = \int_0^x f_1[x_1] dx_1 + \int_0^y f_2[x_2] dx_2 \]  \hspace{1cm} (4.52c)

It can be noted that the functions \( V_1[\cdot], V_3[\cdot] \) are positive definite whereas \( V_2[\cdot] \) is positive semi-definite (becomes zero for some \( x_1 \neq 0 \)). The derivatives of the Lyapunov function candidates along the system trajectories are:

\[ \dot{V}_1[x_1] \bigg|_{x_1=x_0} = \frac{\partial V_1}{\partial x_1} f_1[x_1] \]

\[ = \begin{bmatrix} b_1 x + c_1 x^3 \\ x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ -b_1 x - c_1 x^3 - a_1 \tan^{-1}[x_1] \end{bmatrix} \]  \hspace{1cm} (4.53a)

\[ = (b_1 x + c_1 x^3)x_1 + x_1 \{-b_1 x - c_1 x^3 - a_1 \tan^{-1}[x_1]\} \]

\[ = -a_1 x_1 \tan^{-1}[x_1] \Rightarrow \begin{cases} < 0 & \text{for } x_1 
eq 0 \\ = 0 & \text{for } x_1 = 0 \end{cases} \]

\[ \dot{V}_2[x_2] \bigg|_{x_2=x_0} = \frac{\partial V_2}{\partial x_2} f_2[x_2] \]

\[ = \begin{bmatrix} b_2 \sin[x] \\ x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ -b_2 \sin[x] - a_2 x_1 \end{bmatrix} \]  \hspace{1cm} (4.53b)

\[ = -a_2 x_1 \Rightarrow \begin{cases} < 0 & \text{for } x_1 
eq 0 \\ = 0 & \text{for } x_1 = 0 \end{cases} \]

\[ \dot{V}_3[x_3] \bigg|_{x_3=x_0} = \frac{\partial V_3}{\partial x_3} f_3[x_3] \]

\[ = \begin{bmatrix} f_1[x_1] \\ f_2[x_2] \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} f_1[x_1] \\ f_2[x_2] \end{bmatrix} \]  \hspace{1cm} (4.53c)

\[ = -f_2[x_2] f_3[x_2(t)] \Rightarrow \begin{cases} < 0 & \text{for } x_2 
eq 0 \\ = 0 & \text{for } x_2 = 0 \end{cases} \]

Note that the derivatives of the Lyapunov function candidates are negative semi-definite (and not negative definite since they become zero for some non-zero values of the state variables. For example, for system 1, the derivative is zero for all values of the first state variable \( x(t) \) when the other state variable \( \dot{x} = 0 \)).

---

\(^{13}\) When we select a scalar function \( V[x] \), we do not know whether it will really have the properties of a Lyapunov function and hence the term LF candidate. In chapter 6, we will learn how to select LFs using electrical network equivalence.
Since $\dot{V}_i[.]$ \((i=1\ to\ 3)\) evaluated on the system trajectory is negative semi-definite, it is clear that $\dot{V}_i[.]$ will be a monotonically non-increasing function of time $t$. If it were negative definite, we can infer that $\dot{V}_i[.]$ will be a monotonically decreasing function of time, will go to zero as $t$ goes to infinity, and stay there. Thus $x = \dot{x} = 0$ will be a stable equilibrium point and the system will be asymptotically stable\(^4\). However, as the derivative can be zero (negative semi-definite) at times, there is the possibility that $\dot{V}_i[.]$ will settle at a non-zero value and stay there permanently unless we can prove to the contrary. We will show that $\dot{V}_i[.]$ will continue to decrease using example 1. From $V_i[.]$ and $\dot{V}_i[.]$, we note that for $\dot{x}_i = [k\ 0]^T$ with $k \neq 0$:

\[
V_i[x_i = \dot{x}_i = \begin{bmatrix} k \neq 0 \\ 0 \end{bmatrix} = \frac{b}{2} k^2 + \frac{c}{4} k^4 > 0 \quad (4.54a)
\]

indicating that $\dot{x}_i = [k\ 0]^T$ with $k \neq 0$ is a possible stable point. However, from the nonlinear dynamics (4.51a), we can note that for $x_i = \dot{x}_i$, we have $\dot{x}_i = [0\ k]^T$ with $k \neq 0$. This implies $x_i$ will change and the system will move out of that point immediately. Combining this with the property that $\dot{V}_i[.]$ is negative semi-definite implies that the system will move to a new state with lower residual energy. That is, the trajectory has to move towards the equilibrium point $[0, 0]$. Thus, it is sufficient (and of course necessary) for $\dot{V}_i[.]$ to be negative semi-definite as long as the equilibrium point $x_e$ belongs to the set of points for which $\dot{V}[x] = 0$.

The mathematical tool (known as invariant set theorem) for a vigorous proof of the above statement and the stability concept can be found in other texts and is not really needed at this level. It is sufficient to know that a scalar function of the state variables has to be formed and tested for the LF conditions.

### 4.4 Forced Response of Nonlinear Autonomous Systems

Recall, that the various concepts such as equilibrium points, stability of equilibrium points, Lyapunov functions and so on that we studied in the last section are based on the response of a nonlinear dynamics in the form of a nonlinear differential equation for some initial values & with no forcing function or the transient response as is commonly known in LTI system theory. In practical situations, we need to consider not just the response of systems for initial conditions, but also for external inputs or forcing functions. Again, the two types of inputs or forcing functions, constant valued functions and sinusoidal functions, that are used in deterministic LTI system theory will be used here as well and the response of nonlinear systems for such inputs will be studied in this section. However, the input can appear in complex ways in the nonlinear dynamics, as we will see next. Thus, once again we are faced with the problem that a closed-form solution cannot be found for all classes of nonlinear differential equations. Therefore, we will look at the forced response in general terms and from some specific nonlinear differential equations and derive useful information from them.

#### 4.4.1 Forcing Functions in Nonlinear Dynamics

In general, the presence of forcing functions is indicated by the inclusion of the input vector $u$ in the state model as:

\[
\dot{x} = f[x, u, t] \quad (4.56)
\]

The implications from such a general representation are enormous. Without any guidance from real-world plants, we can come up with exotic models where the inputs get transformed to complex form and or get modulated (or multiplied) by complex functions of the state variables. This is normally the case in the classical approach to nonlinear dynamics. Considering the single input case, the various ways (in the order of increasing complexity) that the input can appear in the dynamics are:

1) Input added additively to just one state equation. That is, we have:

\[
\dot{x} = f[x, t] + bu \quad (4.57)
\]

where the vector $b$ has just one constant non-zero entry.

2. Input added additively to more than one state equation. That is, in the above state model, there will be more than one constant non-zero entry in the vector $b$.

3. Sum of weighted, differentiated input is added to the state equations. That is:

\[
\dot{x} = f[x, t] + B \begin{bmatrix} u(t) \\ du(t)/dt \\ d^2u(t)/dt^2 \\ \vdots \\ d^M u(t)/dt^M \end{bmatrix} \quad (4.58)
\]
where \( B \) is a constant matrix of appropriate dimensions. Of course, this is possible only if the input is a smoothly varying function with finite valued derivatives.

4. The input multiplied by a function of the state variables is added to the state model. Since numerous combinations and permutations are possible, most of the research is based on what is known as canonical form:

\[
\dot{x} \equiv \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} f(x) \\ \vdots \end{bmatrix} + u(t) \tag{4.59}
\]

where \( f[x] \) and \( g[x] \) are scalar functions of the state. In the canonical model, we restrict the state model (without the input) as well as can seen from the first vector on the right hand side of the above equation. Note that this model corresponds to what is known as direct form structure with all the nonlinearities lumped into the feedback for the derivative of the last state vector \( x_n(t) \).

It is obvious from the above models that not much physical insight has gone into the selection of the forms of these models. The first three models are straightforward extensions of the LTI case and can be justified from electrical circuits’ perspective. However, the fourth model, a nonlinear model involving the input, is a simple extension of the LTI canonical model and doesn’t point to valid electrical circuits (as we will see in later chapters). We will address these issues further in the section on Nonlinear System Modeling and Control.

### 4.4.2 Is separation into Transient and Forced Response Necessary?

We first need to address if separation into transient response and forced response is necessary as is common in LTI system theory and not so in the nonlinear arena. In fact, as we indicated in the last section (while discussing Lyapunov direct approach for stability analysis of nonlinear dynamics), the conventional wisdom is that the stability of nonlinear dynamics given either initial state values or constant forcing functions can be treated under the same umbrella. The rationale for this approach is that a nonlinear dynamics with forcing functions \( u(t) \) denoted as:

\[
\dot{x} = f[x, t, u] \tag{4.60}
\]

with the equilibrium point \( x^*(t) \) can be converted to an equivalent dynamics with out any forcing functions as:

\[
\dot{e} = \hat{f}[e, t] \tag{4.61}
\]

where \( e = x - x^* \) is the error vector. The stability of the equilibrium point, \( e = 0 \), of this new dynamics can then be studied using the methods for the transient response. Though this argument is perfectly valid from a mathematical perspective, it destroys the connection between the nonlinear dynamics and the physical system that it represents (We will demonstrate this in chapter # 6 with some examples from nonlinear passive networks).

It can be also argued that the reason for treating the transient response and the forced response separately in LTI systems is the applicability of the superposition theorem that allows us to obtain the responses individually and combine them together. That is definitely not the case here. However, again, the physical considerations dictate that the cases be dealt with separately as we will see in chapter # 6.

### 4.4.3 BIBO Stability and Total Stability of Nonlinear Autonomous Systems

Given a nonlinear dynamics that is asymptotically stable, we are interested in knowing what kinds of responses are possible from such a dynamics when a constant or periodic forcing function is added. In the case of LTI systems, we know that an absolutely stable LTI system will also give rise to bounded outputs when excited by a bounded input, leading to the notion of BIBO stability. Thus, we can ask if there is a similar notion of stability in the case of nonlinear dynamics.

Of course, as we noted in section 4.4.1, the application of the input and its effect on the nonlinear dynamics itself are complex issues. Thus, we need to define clearly how the input appears in the state model. Luckily for us, even the simplest additive model given in equation (4.57) leads to some interesting results not seen in LTI systems. We will now make some useful observations and conclusions based on some simple examples. Consider two nonlinear systems given by:

\[
\dot{x} + \tanh[x] = f_1[t] \tag{4.62a}
\]

\[
\ddot{x} + \tanh[\dot{x}] + x = f_2[t] \tag{4.62b}
\]
Setting \( f_1(t) = f_2(t) = 0 \), we can note that \( x = 0 \) and \( x = [x \ ẋ = x_1]^T = [0 \ 0]^T \) respectively are the equilibrium points of the two systems. Choosing Lyapunov function candidates as:

\[
V_1[x] = \frac{1}{2} x^2 \tag{4.63a}
\]

\[
V_2[x] = \frac{1}{2} (x^2 + x_1^2) \tag{4.63b}
\]

which are positive definite, the derivatives of the Lyapunov candidates along the system trajectory are given by:

\[
\dot{V}_1[x] = -x \tanh[x] \tag{4.64a}
\]

\[
\dot{V}_2[x] = -\dot{x} \tanh[\dot{x}] = -x_1 \tanh[x_1] \tag{4.64b}
\]

That is, the derivative of the Lyapunov function candidates for system # 1 is negative definite indicating that the equilibrium point is stable. For system # 2, we find the derivative is only negative semi-definite and becomes zero for \( \dot{x} = [k \ 0]^T \) with \( k \neq 0 \) or on the entire \( x \)-axis. Since the equilibrium point \([0, 0]\) falls on this set, we can conclude using arguments seen before that this equilibrium point is stable. Since there is only one equilibrium point for both systems, and the corresponding Lyapunov functions are radially unbounded, we can also conclude that the systems are globally, asymptotically stable. In fact, we can note that the equilibrium point will also be exponentially stable as the dynamics approximates a stable linear dynamics near the equilibrium point.

Considering the case of \( f_1(t) = f_2(t) = v_{DC} \), where \( v_{DC} \) is a constant, we find that the response \( x(t) \) of the first-order system will be given by:

\[
x_{DC} = x(t)|_{t=\infty} = \tanh^{-1}[v_{DC}] \text{ if } |v_{DC}| < 1 \tag{4.65a}
\]

and the response of the second-order system will be:

\[
x_{DC} = x(t)|_{t=\infty} = v_{DC} \tag{4.65b}
\]

Thus, we get a constant valued response in both cases as the steady state response. We can call this an equilibrium point as we are dealing with a response that is a point in the \( N \) dimensional state space. Using a change of variables, \( e = x - x_{DC} \), we can re-write the two dynamics with the constant forcing function as:

\[
\dot{e} + f[e] = 0 \text{ where } f[e] = \tanh[e + x_{DC}] - v_{DC} \tag{4.66a}
\]

and

\[
\dot{\tilde{e}} + \tanh[\dot{\tilde{e}}] + e = 0 \tag{4.66b}
\]

where \( f[e] \) is a function of the new variable \( e(t) \). We can note that the two new dynamics with the origin as the equilibrium point are asymptotically stable and hence the equilibrium of the two original dynamics under constant excitation will also be stable. That is the DC response will go to and settle at the value \( x_{DC} \) and stay close to it if any disturbance moves the state away from the equilibrium point.

Of course, a finite, constant valued solution is possible for the given first-order dynamics only if \( |v_{DC}| < 1 \). Thus, we cannot talk about the response to bounded inputs where the bound is almost unlimited as we did in the case of LTI systems and BIBO stability is not a real valid term for nonlinear dynamics. Hence, we have to restrict our study of the forced response of the dynamics for external input or disturbance of limited amplitude. If the response stays close to the equilibrium point and returns to the equilibrium point when the disturbance is removed, such systems are known as "Totally stable nonlinear systems" rather than of BIBO stable systems.

In the above examples, the state will reach the value \( x_{DC} \) regardless of the initial condition. This is not necessarily true for all nonlinear systems, as the next example will illustrate. Consider the nonlinear dynamics given by:

\[
\dot{x} + x(x^2 - 1) = v_{DC} \tag{4.67}
\]

We can note that this nonlinear dynamics \( v_{DC} = 0 \) has three equilibrium points, \( x = -1, 0, 1 \), of which -1 and +1 are stable and the origin is unstable. If we apply a constant excitation of value \( v_{DC} = 0.25 \), we find that three solutions, \( x_{DC} = -1.10724, 0.2644, 0.834 \), are possible as shown in Fig. 4.15. We can easily show that the two solutions, \( x_{DC} = -1.10724, 0.834 \), are locally stable, and one of which is reached depending on the initial condition. Also, disturbances will (or will not) make the equilibrium jump from one equilibrium point to another depending on the amplitude. Thus, we have stable equilibrium points which are stable from a totally stable system perspective and not from the BIBO perspective.
Let us now consider the above two nonlinear systems with the \( \tanh \) nonlinearity driven by a periodic source, the simplest being a sinusoidal one given by:

\[
f_1(t) = f_2(t) = A \sin(t)
\]

where \( A \) is the amplitude that we can vary. Both systems have only one nonlinear term, \( \tanh[x] \) or \( \tanh[\dot{x}] \) which becomes linear for \( x \) or \( \dot{x} \approx 0 \) (see fig. 4.16 for the actual curve and its linear approximation). Thus, when \( A \) is very small (\( |A| < 0.1 \)), both systems act like a linear system with the approximate responses taking the form:

\[
x(t) = A_1 \sin(t + \theta_1)
\]

\[
x(t) = A_2 \sin(t + \theta_2)
\]

\[
x_1(t) = A_3 \cos(t + \theta_3)
\]

where \( A_1, \ A_2, \text{ and } A_3 \) are finite constants. That is, the output is bounded when the input is bounded (and of very small amplitude). Thus, we find both systems that are globally, asymptotically stable also produce bounded outputs not just for constant excitation but also when subjected to sinusoidal signals of small amplitude. However, for larger signal amplitudes the situation is different. A closed form solution for the forced response of the second system is given by:

\[
x(t) = \frac{A_2}{2} (\sin(t) - t \cos(t)) - \int_0^t \sin(t - \tau) \tanh(\dot{x}(t)) \, d\tau \geq \frac{A_2}{2} (\sin(t) - t \cos(t)) - \int_0^t |\sin(t - \tau)| \, d\tau
\]

Therefore, we find that for large \( A \), the responses \( x(t) \) and \( x_1(t) \) go to infinity. The same thing happens with the output of the first system. That is, we find that we can have nonlinear systems that are globally, asymptotically stable which may not necessarily be BIBO stable in the true LTI systems perspective.

The culprit here is the saturating nonlinearity, \( \tanh[x] \) or \( \tanh[\dot{x}] \), which represents the dissipation in the circuits corresponding to the dynamics and is limited in its ability to consume the energy that is being injected into the circuit by the independent source. Thus, the unspent energy gets stored in the storage.
elements making the state variables to grow with out bound. We will look at this issue more detail in chapter 6.

In summary, we find that the inputs can appear in exotic forms in the nonlinear dynamics. Even when it appears in a simpler form, the forced response of nonlinear dynamics that are asymptotically stable may or may not be bounded for all magnitudes of the input and a knowledge of the bounds on the input(s) is necessary. In later chapters, we will use nonlinear electrical networks' concepts to arrive at such bounds.

### 4.5 Summary

In this chapter we went through autonomous nonlinear systems from a classical, mathematical perspective. We noted that time-domain representation, especially the state space representation, is the most general and preferred form for representing such systems. We learnt that due to the presence of nonlinear terms in the model, nonlinear systems can have multiple equilibrium points, some or all of that can be stable or unstable. Also, the presence of multiple equilibrium points require that we use modifiers such as global, local etc. to characterize the stability of equilibrium points. We also found that a system may be asymptotically stable, but the response may take long time to reach the equilibrium point necessitating a restricted category of asymptotically stable systems, that of exponentially stable systems. We observed that the Lyapunov's direct method for checking the stability of an autonomous nonlinear system perhaps evolved by considering the energy left at any time in an otherwise undisturbed [by an external input] second-order physical system with initial stored energy and the procedure later mathematically extended to higher order systems. Further, we learnt that the Lyapunov's direct method provides only a sufficiency condition in that the existence of a Lyapunov function indicates that the system is stable and the converse, our inability to come up with a suitable Lyapunov function doesn't necessarily mean that the system is not stable.

For reasons that will become clear in the next two chapters, we considered the response of nonlinear systems to stored initial energy as different from the response to forced functions, especially DC excitation. This approach, we noted, is different from the approach taken by other researchers in this area. We observed that asymptotic stability that is defined on the basis of the response to stored initial energy doesn't necessarily imply stability under the presence of bounded forced excitation as is the case under LTI systems. Again, a discussion of the physical reasons for such different behavior has been deferred to latter chapters.