Hwk 5 QC

Winter 2012

Due Monday Feb. 20

① Using the fact that $[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$
show that $[L_x, L_y] = i\hbar L_z$
where (L_x, L_y, L_z) are the components of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$, the crossproduct of $\vec{r} = (x, y, z)$ and $\vec{p} = (p_x, p_y, p_z)$.

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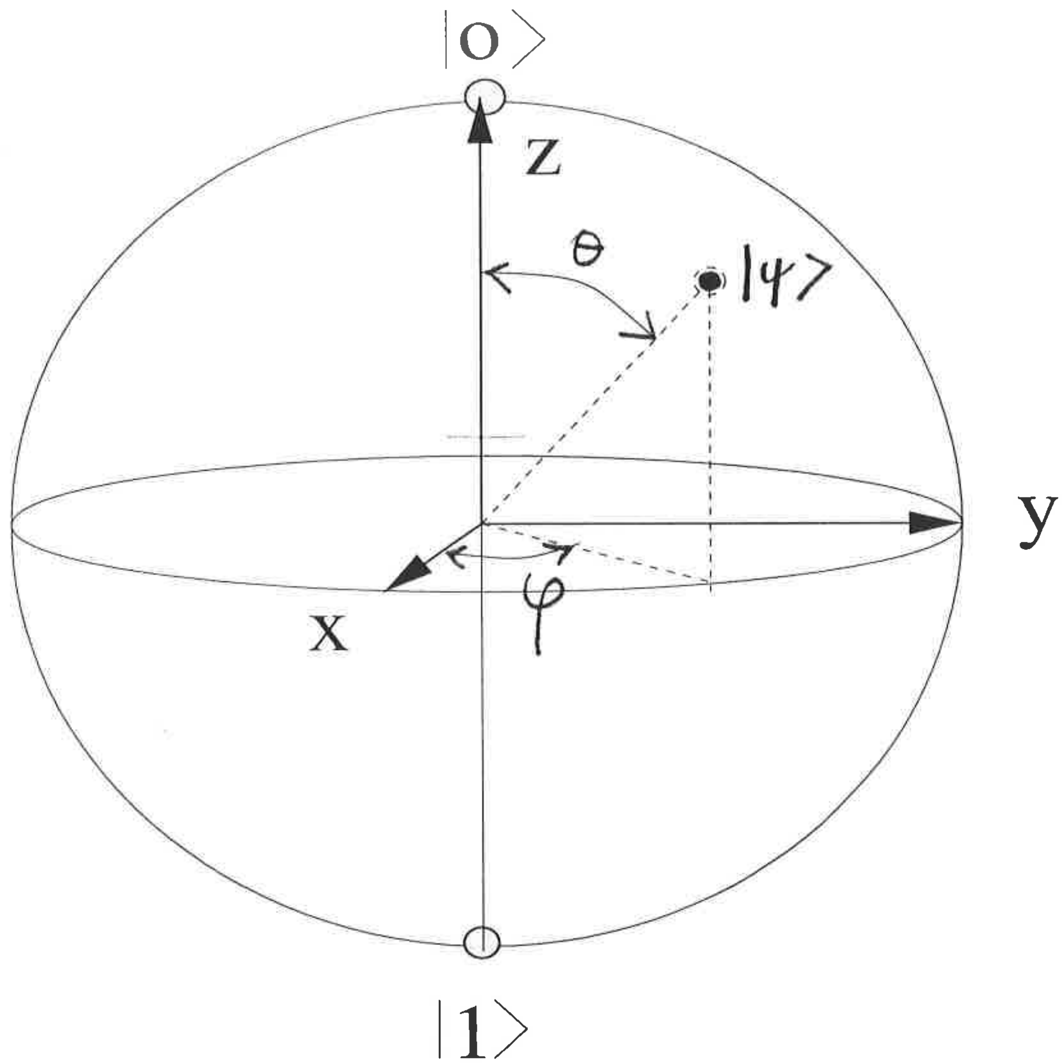
② Consider the qubit

$$|\chi\rangle = \sqrt{\frac{3}{5}}\,|0\rangle - \sqrt{\frac{2}{5}}\,|1\rangle$$

For this qubit, calculate the values of the angles γ, θ, φ associated to the corresponding spinor

$$|\xi_n^+\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}\, e^{i\varphi}|1\rangle\right]$$

Show the location of the spinor on the attached Bloch sphere.



Bloch Sphere Representation of a Qubit

(3) Express the Hadamard gate as a product of rotation matrices $R_y(\theta)$ and $R_z(\theta')$ given in class (specify the angles θ and θ') and some phase factor $e^{i\delta}$. Specify δ.

(4) Show that any 2×2 M matrix can be expressed as follows

$$M = a_0 \mathbb{I} + \vec{a} \cdot \vec{\sigma}$$

where $$a_0 = \frac{1}{2} \mathrm{Tr}\, M$$

and $$\vec{a} = \frac{1}{2} \mathrm{Tr}\,(M \vec{\sigma})$$

with $\vec{a} \cdot \vec{\sigma} = a_x \sigma_x + a_y \sigma_y + a_z \sigma_z$

(5) Using the results of the previous exercise, find the values of a_0 and $\vec{a}$ when M is equal to the Phase shift matrix

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

(6) Consider two matrices M_1 and M_2 with the decomposition

$$M_1 = a_0^1 \, \mathbb{1} + \vec{a_1} \cdot \vec{\sigma}$$
$$M_2 = a_0^2 \, \mathbb{1} + \vec{a_2} \cdot \vec{\sigma}$$

Use the identity

$$(\vec{\sigma} \cdot \vec{a_1})(\vec{\sigma} \cdot \vec{a_2}) = \vec{a_1} \cdot \vec{a_2} \, \mathbb{1} + i \vec{\sigma} \cdot (\vec{a_1} \times \vec{a_2})$$

where $\vec{a_1} \times \vec{a_2}$ is the cross-product of the two vectors $\vec{a_1}$ and $\vec{a_2}$.

What must be the relationship between $\vec{a_1}$ and $\vec{a_2}$ for the matrices M_1 and M_2 to commute?

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(7) Using the fact that $[\sigma_x, \sigma_y] = 2i\sigma_z$

Calculate the commutator $[R_x(\theta_1), R_y(\theta_2)]$

where
$$R_x(\theta_1) = e^{-i\frac{\theta_1}{2}\sigma_x} \; ; \quad R_y(\theta_2) = e^{-i\frac{\theta_2}{2}\sigma_y}$$

Hint: Use the fact that

$$e^{i\alpha A} = \cos\alpha \, I + \sin\alpha \, A,$$ where I is the 2×2 identity matrix.