60/60

Intro to Quantum Computing : ECES622 - Winter 2012

Homework 4

3 February 2012

Hua Li, Kevin Pasko, Allen Welch

(I) Prove that

$$\sigma_x \sigma_y \sigma_z = iI$$

where σ_x, σ_y, and σ_z are the three Pauli matrices and I is the identity matrix.

The Pauli X, Pauli Y and Pauli Z matrices are: $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. And the 2x2 identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\begin{aligned}
\sigma_x \sigma_y \sigma_z &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times 0 + 1 \times i & 0 \times -i + 1 \times 0 \\ 1 \times 0 + 0 \times i & 0 \times -i + 1 \times 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} i \times 1 + 0 \times 0 & i \times 0 + 0 \times -1 \\ 0 \times 1 + -i \times 0 & 0 \times 0 + -i \times -1 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \\
&= i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= iI
\end{aligned}$$

10/10

(II) If a spin is in the qubit state (which is properly normalized)

$$|\Psi\rangle = \frac{1}{\sqrt{10}}(3|0\rangle - |1\rangle)$$

what is the probability to find a value of $\frac{\hbar}{2}$ if you measure the y-component of the spin?

10/10

$$|\Psi\rangle = \frac{1}{\sqrt{10}}(3|0\rangle - |1\rangle) = \frac{1}{\sqrt{10}}(3\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}) = \frac{1}{\sqrt{10}}\begin{bmatrix}3\\-1\end{bmatrix}$$

$$S_y = \left(\frac{\hbar}{2}\right)\sigma_y = \left(\frac{\hbar}{2}\right)\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = \begin{bmatrix}0 & -\left(\frac{\hbar}{2}\right)i\\ \left(\frac{\hbar}{2}\right)i & 0\end{bmatrix}$$

$$det(S_y - \lambda I) = \begin{vmatrix}-\lambda & -\left(\frac{\hbar}{2}\right)i\\ \left(\frac{\hbar}{2}\right)i & -\lambda\end{vmatrix} = 0$$

$$Eigenvalues\colon\ \lambda_1 = -\frac{\hbar}{2}, \qquad \lambda_2 = \frac{\hbar}{2}$$

$$\lambda_1 = -\frac{\hbar}{2} \to (S_y - \lambda I)|-\rangle_y = 0 \to \begin{bmatrix}\frac{\hbar}{2} & -\left(\frac{\hbar}{2}\right)i\\ \left(\frac{\hbar}{2}\right)i & \frac{\hbar}{2}\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = 0 \to \begin{cases}a = 1\\ b = -i\end{cases}$$

$$Normalized\ Eigenvector\colon |-\rangle_y = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$$

$$\lambda_2 = \frac{\hbar}{2} \to (S_y - \lambda I)|+\rangle_y = 0 \to \begin{bmatrix}-\frac{\hbar}{2} & -\left(\frac{\hbar}{2}\right)i\\ \left(\frac{\hbar}{2}\right)i & -\frac{\hbar}{2}\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = 0 \to \begin{cases}a = 1\\ b = i\end{cases}$$

$$Normalized\ Eigenvector\colon |+\rangle_y = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$$

$$P(m) = \langle\Psi|P_m|\Psi\rangle = |\langle m|\Psi\rangle\,|^2$$

$$P\left(\frac{\hbar}{2}\right) = |\langle m|\Psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}[1 \quad -i]\frac{1}{\sqrt{10}}\begin{bmatrix}3\\-1\end{bmatrix}\right|^2 = \left|\frac{1}{\sqrt{20}}(3+i)\right|^2 = \frac{1}{20}(9+1) = \frac{1}{2}$$

(III) Show that the matrix $\sqrt{\text{NOT}}$ given by

$$\begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

is unitary.

For $\sqrt{\text{NOT}}$ to be unitary, the following statement must be true:

$$\sqrt{\text{NOT}}^{\dagger}\sqrt{\text{NOT}} = \sqrt{\text{NOT}}\sqrt{\text{NOT}}^{\dagger} = I_2$$

Where $\sqrt{\text{NOT}}^{\dagger}$ is the Hermitian conjugate of $\sqrt{\text{NOT}}$:

10/10

$$\sqrt{\text{NOT}}^{\dagger} = \begin{bmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

First, it is shown that $\sqrt{\text{NOT}}^{\dagger}\sqrt{\text{NOT}} = I_2$:

$$\begin{aligned}
\sqrt{\text{NOT}}^{\dagger}\sqrt{\text{NOT}} &= \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}\begin{bmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \\
&= \frac{1}{2} \times \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} (1+i)(1-i) + (1-i)(1+i) & (1+i)(1+i) + (1-i)(1-i) \\ (1-i)(1-i) + (1+i)(1+i) & (1-i)(1+i) + (1+i)(1-i) \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 2+2 & 2i-2i \\ -2i+2i & 2+2 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I_2
\end{aligned}$$

And $\sqrt{\text{NOT}}\sqrt{\text{NOT}}^{\dagger} = I_2$:

$$\sqrt{\text{NOT}}\sqrt{\text{NOT}}^{\dagger} = \begin{bmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$= \frac{1}{2} \times \frac{1}{2} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} (1-i)(1+i) + (1+i)(1-i) & (1-i)(1-i) + (1+i)(1+i) \\ (1+i)(1+i) + (1-i)(1-i) & (1+i)(1-i) + (1-i)(1+i) \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 2+2 & -2i+2i \\ 2i-2i & 2+2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I_2$$

$\sqrt{\text{NOT}}^{\dagger}\sqrt{\text{NOT}} = I_2$ and $\sqrt{\text{NOT}}\sqrt{\text{NOT}}^{\dagger} = I_2$, thus $\sqrt{\text{NOT}}^{\dagger}\sqrt{\text{NOT}} = \sqrt{\text{NOT}}\sqrt{\text{NOT}}^{\dagger} = I_2$ is true and $\sqrt{\text{NOT}}$ is unitary.

(IV) What is the 4x1 column vector resulting from the following tensor product

$$(\mathrm{H} \otimes \sigma_x)(|0\rangle \otimes |1\rangle)$$

where H is the Hadamard matrix

$$\mathrm{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$(\mathrm{H} \otimes \sigma_x)(|0\rangle \otimes |1\rangle) = \left(\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

10/10

$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (0 \times 0) + (\frac{1}{\sqrt{2}} \times 1) + (0 \times 0) + (\frac{1}{\sqrt{2}} \times 0) \\ (\frac{1}{\sqrt{2}} \times 0) + (0 \times 1) + (\frac{1}{\sqrt{2}} \times 0) + (0 \times 0) \\ (0 \times 0) + (\frac{1}{\sqrt{2}} \times 1) + (0 \times 0) + (-\frac{1}{\sqrt{2}} \times 0) \\ (\frac{1}{\sqrt{2}} \times 0) + (0 \times 0) + (-\frac{1}{\sqrt{2}} \times 0) + (0 \times 0) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(V) Compute the following Kronecker product $H \otimes H \otimes H$ where H is the Hadamard Matrix.

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{aligned} H \otimes H \otimes H &= H \otimes (H \otimes H) \\ &= H \otimes \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \\ &= \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \end{aligned}$$

10/10

(VI) What is the average value of the tensor product

$$H \otimes \sigma_z$$

(where H is the Hadamard matrix and σ_z is the Pauli matrix) in the 2-qubit state given by

$$[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle] \otimes [\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)]$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

10/10

$$H \otimes \sigma_z = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)] \otimes [\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)]$$

$$= \left[\frac{1}{\sqrt{2}}([1 \quad 0] - [0 \quad 1])\right] \otimes \left[\frac{1}{\sqrt{2}}([1 \quad 0] + [0 \quad 1])\right] = \left[\frac{1}{\sqrt{2}}[1 \quad -1]\right] \otimes \left[\frac{1}{\sqrt{2}}[1 \quad 1]\right]$$

$$= \frac{1}{2}[1 \quad 1 \quad -1 \quad -1]$$

$$[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle] \otimes [\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)]$$

$$= \left[\frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right] \otimes \left[\frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right] = \left[\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right] \otimes \left[\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\langle \Psi | [M] | \Psi \rangle$$

$$= \left\langle \left[\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\right] \otimes \left[\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\right] \middle| [H \otimes \sigma_x] \middle| [\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle] \otimes [\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)] \right\rangle$$

$$= \frac{1}{2 \times 2 \times \sqrt{2}}[1 \quad 1 \quad -1 \quad -1]\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$= 0$$