There is a useful way of representing linear operators which makes use of the inner product.

Take \( |w> \) in inner product space \( V \)

\[ |w> = \sum_i \alpha_i |v_i> \]

Define \( |w>_\wedge \wedge |v> \) to be the linear operator from \( V \rightarrow W \) such that \( \forall e \in V \)

\[ (|w>_\wedge \wedge |v>) \equiv |w> \otimes |v> = |w_\wedge v> = \sum_i \alpha_i |v_i> |w> \]

Linear combination of outer products:

\[ (\sum_i \alpha_i |v_i> \otimes |v>) = \sum_i \alpha_i |v_i> |v> \]

Usefulness of outer product notation:

Consider an orthonormal basis set \( \{ i > \} \)

\[ |v> = \sum_i \alpha_i |i> \]

\[ <i|v> = \alpha_i \]

So \( (\sum_i \alpha_i |i> \otimes |i>) \equiv \sum_i \alpha_i |i> \otimes |i> = \sum_i \alpha_i |i> |i> = |v> \]

This is true for all \( |v> \)'s in \( V \)!

\[ \sum_i |i> \otimes |i> = V \]

This is known as the Completeness relation

Problem: check it is true for bases \( [10> \otimes 11> \) and \( [11> \otimes 12> \)

Note: \( d \) is dimension of \( V \)
First use of Completeness relation

Representation of an operator in the outer product notation

Suppose we have a linear operator \( A : V \rightarrow W \)

\( |w_i\rangle \) is an orthonormal basis in \( V \)

\( |w_j\rangle \) is \( |w_i\rangle \) \( \forall i \leq n \)

Then

\[
A = I_W A I_V
\]

So,

\[
A = \sum_j |w_j\rangle \langle w_j| A \sum_i |w_i\rangle \langle w_i|
\]

\[
A = \sum_j \langle w_j| A |w_j\rangle |w_j\rangle \langle w_j|
\]

which is called the outer product representation of \( A \).

Exercise 2.9 P.68

Pauli operators and the outer product. \( V = W = \mathbb{C}^2 \)

The Pauli matrices can be considered as operators with respect to an orthonormal basis \( |0\rangle, |1\rangle \)
for a two-dimensional Hilbert space.

Express each of the Pauli matrices in the outer product notation \((\mathbf{F}, \mathbf{G}, \mathbf{H})\)
\[ \sigma_x = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{&} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ \sigma_x = \sum_{j=0}^{\infty} \langle i | \sigma_x | j \rangle |i\rangle |j\rangle \]

\[ \langle 0 | \sigma_x | 1 \rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \]

\[ \langle 0 | \sigma_x | 0 \rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \]

\[ \langle 1 | \sigma_x | 0 \rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \]

\[ \langle 1 | \sigma_x | 1 \rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \]

\[ \Rightarrow \sigma_x = (-1) \langle 0 \rangle \langle 1 \rangle + 0 \langle 0 \rangle \langle 0 \rangle + 1 \langle 1 \rangle \langle 0 \rangle + 0 \langle 1 \rangle \langle 1 \rangle \]

\[ \sigma_x = \langle 0 \rangle \langle 1 \rangle + \langle 1 \rangle \langle 0 \rangle \]

The outer product representations of \( \sigma_y \) and \( \sigma_z \) can be derived the same way.

Note: The outer product representation would be different if we had selected another orthonormal base in \( \mathbb{C}^2 \).

As an exercise, you could repeat the analysis above using the basis \( |\psi_1\rangle, |\psi_2\rangle \) on page 4.
\[ \| \vec{v} \cdot \vec{w} \| \leq \| \vec{v} \| \| \vec{w} \| \cos \theta \]

\[ -1 \leq \cos \theta = \frac{\vec{v} \cdot \vec{w}}{\| \vec{v} \| \| \vec{w} \|} \leq 1 \]
Second application of the completeness relation

The Cauchy-Schwarz inequality

\[ \forall \mid \psi \rangle, \mid \phi \rangle \in \mathbb{V} \quad (\text{dimension } d) \]

\[ |\langle \psi | \phi \rangle|^2 \leq \langle \psi | \psi \rangle \langle \phi | \phi \rangle \]

We use the Gram-Schmidt procedure to construct an orthonormal basis of vectors \( \mid i \rangle \) with the first \( \mid i \rangle \) as

\[ \mid i \rangle = \frac{\mid \omega \rangle}{\| \mid \omega \rangle \|} = \frac{\mid \omega \rangle}{\sqrt{\langle \omega | \omega \rangle}} \]

Using \( \sum_i \mid i \rangle \langle i \mid = \mathbb{I} \) (Completeness relation)

\[ \langle \psi | \phi \rangle = \sum_i \langle \psi | i \rangle \langle i | \phi \rangle \]

Dropping some non-negative terms (keeping only \( i = 1 \) term)

\[ \langle \phi | \psi \rangle \geq \left[ \frac{\langle \psi | \phi \rangle}{\sqrt{\langle \phi | \phi \rangle}} \right] \langle \psi | \phi \rangle \]

Hence \( \langle \phi | \psi \rangle \leq \langle \psi | \phi \rangle \)

or \( \| \psi \|^2 \leq \langle \psi | \psi \rangle \langle \phi | \phi \rangle \)

Equality occurs if \( \mid \psi \rangle = \beta \mid \phi \rangle \), i.e., if the 2 vectors are linearly related (i.e.,
Section 2.1.5 Eigenvalues and Eigenvectors

Definition: An eigenvector of a linear operator $A$ is a non-zero vector $|v>$ such that

$$A|v> = \lambda |v>$$

where $\lambda$ is a complex number known as the corresponding eigenvalue.

Eigenvalues $\lambda$ are solutions of

$$\det |A - \lambda I| = 0$$

a polynomial in $\lambda$

The eigenspace corresponding to an eigenvalue $\lambda$ is the set of vectors with eigenvalue $\lambda$. It is a vector subspace of the vector space on which $A$ acts.

Diagonal representation of $A$ also known as Orthogonal Decomposition of $A$

$$A = \sum \Lambda_i |i><i|$$

where $|i>$ vectors form an orthonormal set of eigenvectors of $A$.

An operator is diagonalizable if it has a diagonal representation. Soon, we will see the necessary and sufficient conditions for an operator to be diagonalizable.
Problem 1: Consider the following matrix $A$

\[
\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix}
\]

(1) • Calculate the eigenvalues and corresponding eigenvectors of this matrix. Normalize the eigenvectors.

(2) • What are the angles $\theta$ and $\phi$ in the general expression of the qubit on the Bloch sphere, i.e.,

\[
|\xi_0^+\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle,
\]

associated to the two eigenvectors of the matrix $A$?

(3) • Give the outer product representation of $A$ using the basis formed by the two kets $|0\rangle$ and $|1\rangle$, i.e., what are the coefficients ($\alpha$, $\beta$, $\gamma$, and $\delta$) in the decomposition:

\[
A = \alpha|0\rangle\langle 0| + \beta|0\rangle\langle 1| + \gamma|1\rangle\langle 0| + \delta|1\rangle\langle 1|?
\]

(4) • Give the expression of a non-diagonal matrix which commutes with the matrix $A$. Explain how you obtained that matrix $B$. 

\[
\begin{vmatrix}
1-\lambda & 2 \\
2 & 1-\lambda
\end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 4 = 0 \\
\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4+12}}{2} = 3, -1
\]

\[
\lambda = 3 \Rightarrow \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \binom{a}{b} = \binom{0}{0} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 + \gamma \hat{e}
\]

\[
\lambda = -1 \Rightarrow \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \binom{a}{b} = \binom{0}{1} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1 - \gamma \hat{e}
\]

$A = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\alpha' (10) < \beta' (01) < \gamma' (10) < \delta' (01)$

\[
\begin{pmatrix} 10 < 01 < 11 \end{pmatrix} \Rightarrow \text{commute with } A.
\]
Example \( \Sigma_z = z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) has obviously eigenvalues 1 and -1.

Furthermore \( \Sigma_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = +1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) or \( \Sigma_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = +1/\sqrt{2} \)
\( \Sigma_z \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) or \( \Sigma_z \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1/\sqrt{2} \)

Therefore the diagonal representation of \( \Sigma_z \) is (orthonormal decomposition)
\[
\Sigma_z = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} = |0><0| + (-1)|1><1|
\]

\[
\Sigma_z = |0><0| - |1><1|
\]

Backwards
\[
|0><0| = (\begin{bmatrix} 1 \\ 0 \end{bmatrix}) (\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) \quad \text{(Kronecker product)}
\]
\[
|1><1| = (\begin{bmatrix} 0 \\ 1 \end{bmatrix}) (\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = (\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})
\]

\( z = \Sigma_z \) so indeed \( |0><0| - |1><1| = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) - (\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) \)
\[
\Sigma_z = (\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})
\]
Section 2.1.6 Adjoints and Hermitian Operators

If $A$ is a linear operator on a Hilbert space $V$, there exists a unique operator $A^\dagger$ (\dagger is pronounced dagger) on $V$ such that

$$
\langle \psi | A \phi \rangle = \langle A^\dagger \psi | \phi \rangle
$$

$A^\dagger$ is called the adjoint or Hermitian conjugate of $A$.

If $A, B$ are linear operators on $V$, $A^\dagger, B^\dagger$ their respective adjoints, then $(AB)^\dagger = B^\dagger A^\dagger$

\[\begin{align*}
\langle \psi | A[B|\phi\rangle &= \langle (AB)^\dagger \psi | \phi \rangle \\
\langle \psi | A [B | \phi \rangle &= \langle A^\dagger [B | \phi \rangle \\
&= \langle (B^\dagger A^\dagger) | \phi \rangle \\
&= \langle A^\dagger (B^\dagger) | \phi \rangle
\end{align*}\]

So, indeed $(AB)^\dagger = B^\dagger A^\dagger$

By convention, $|\psi\rangle^\dagger = \langle \psi |$ and $|\phi\rangle = \langle \phi |$

So $\langle A | \phi \rangle^\dagger = \langle \phi | A^\dagger$
\[ A^+ = A \]

**Properties of Adjoint**

1. \((A^+)^+ = A\)
2. \((\sum_i e_i A_i)^+ = \sum_i e_i^* A_i^*\) adjoint operator is antilinear
3. \((1\langle w, v \rangle)^+ = 1\langle w^*, v \rangle \implies 1\langle w, v \rangle \text{ is Hermitian for any vector } 1w\rangle\)

**Matrix Representation of Adjoint**

\[ A^+ = (A^*)^T \]

\(\ast\) stands for complex conjugation

\(T\) is transpose

**Example:**

\[ A = \begin{bmatrix} 1+3i & 2i \\ 1+i & 1-4i \end{bmatrix} \]

\[ A^+ = \begin{bmatrix} 1-3i & -2i \\ -1-i & 1+4i \end{bmatrix}^T = \begin{bmatrix} 1-3i & 1i \\ -2i & 1+4i \end{bmatrix} \]

An operator whose adjoint is A itself is said to be **self-adjoint**

or **Hermitian**

\[ A^+ = A \]

adjacent elements are real
\[(A^+)^+ = A.\]

**Proof**
Call \(B = A^+\)

\[
\langle n\rangle, B^+ | n\rangle \\
\| \\
\langle B | n\rangle, | n\rangle \\
\| \\
\langle A^+ | n\rangle, | n\rangle \\
\| \\
\langle n\rangle, A^+ | n\rangle
\]

Equal \(\Rightarrow B^+ = A \) or 
\[(A^+) = A\]
\[
\sum_i (\mathbf{a}_i) = \sum_i \mathbf{a}_i \cdot \mathbf{A}_i
\]

(2.3b)

First

\[
(\mathbf{A} + \mathbf{B})^+ = \mathbf{A}^+ + \mathbf{B}^+
\]

\[
(1\mathbf{w}^>, (\mathbf{A} + \mathbf{B})_1 \mathbf{w}^>) = ((\mathbf{A} + \mathbf{B})^+_1 \mathbf{w}^>, 1\mathbf{w}^>)
\]

\[
(1\mathbf{w}^>, (\mathbf{A} + \mathbf{B})_1 \mathbf{w}^>) = (\mathbf{A}^+_1 \mathbf{w}^>, 1\mathbf{w}^>) + (\mathbf{B}^+_1 \mathbf{w}^>, 1\mathbf{w}^>)
\]

\[
= ((\mathbf{A} + \mathbf{B})^+_1 \mathbf{w}^>, 1\mathbf{w}^>)
\]

So

\[
(\mathbf{A} + \mathbf{B})^+ = \mathbf{A}^+ + \mathbf{B}^+ \quad \text{indeed}
\]

Second

\[
(\alpha \mathbf{A})^+ = \alpha^* \mathbf{A}^+
\]

\[
(1\mathbf{w}^>, \alpha \mathbf{A}_1 \mathbf{w}^>) = (\alpha^* \mathbf{A}_1 \mathbf{w}^>)
\]

\[
\alpha (1\mathbf{w}^>, \mathbf{A}_1 \mathbf{w}^>) = (\alpha^* \mathbf{A}_1 \mathbf{w}^>)
\]

\[
\alpha (\mathbf{A}^+_1 \mathbf{w}^>, 1\mathbf{w}^>) = (\alpha^* \mathbf{A}^+_1 \mathbf{w}^>)
\]

\[
\alpha (1\mathbf{w}^>, \mathbf{A}_1 \mathbf{w}^>) = (\alpha^* \mathbf{A}_1 \mathbf{w}^>)
\]

\[
\alpha (\mathbf{A}^+_1 \mathbf{w}^>, 1\mathbf{w}^>) = (\alpha^* \mathbf{A}^+_1 \mathbf{w}^>)
\]
\((|n\rangle\langle n|)^{\dagger} = |n\rangle\langle n-1|\)
Important Class of Hermitian Operators

Projectors

Take $W$ a $k$-dimensional subspace of a $d$-dimensional vector space $V$.

Using the Gram-Schmidt procedure, it is possible to construct an orthonormal basis $|1\rangle, \ldots, |d\rangle$ for $V$ (of dimension $d$) such that $|1\rangle, \ldots, |k\rangle$ is an orthonormal basis for $W$.

By definition, $P = \sum_{i=1}^{k} |i\rangle\langle i|$ is the projector onto the subspace $W$.

$|w\rangle\langle w|$ is Hermitian for any vector $|w\rangle$, hence $P$ is Hermitian, i.e., $P^+ = P$ (Prove it!)

By definition, the orthogonal complement of $P$ is the operator $Q = I - P$. $Q$ is a projector onto the vector space spanned by the vectors $|k+1\rangle, \ldots, |d\rangle$.

Completeness relation:

$\sum_{i=1}^{d} |i\rangle\langle i| = \sum_{i=1}^{k} |i\rangle\langle i| + \sum_{i=k+1}^{d} |i\rangle\langle i|$
An operator is normal if $A^+A = AA^+$

Clearly, any Hermitian operator is also normal.

**Important Theorem:** An operator is a normal operator if it is diagonalizable.

This is called the SPECTRAL DECOMPOSITION THEOREM (P. 72)

A matrix is unitary if $UU^+ = I$

Similarly, an operator is unitary if $U^+U = I$

Clearly, $U^+U = I = UU^+$

A unitary operator $U$ is normal and therefore also has a spectral decomposition.

What's so important about unitary operators?

As we will see later, they play a key role in quantum computing as any action on qubits will be using unitary operators.

These operators preserve inner products between vectors $|v\rangle$ and $|w\rangle$, if $U$ is a unitary operator

$\langle v|w\rangle = \langle U|U^+U|v\rangle = \langle U|I|w\rangle = \langle v|w\rangle$

$\langle 0|v\rangle = \langle 1|v\rangle = \langle 0|U^+U|1\rangle = \langle 0|I|w\rangle = \langle 0|v|w\rangle$
Outer Product Representation of any unitary operator $U$

The previous result leads to the following elegant outer product representation of any operator $U$ (unitary).

Take $|\psi_i\rangle$ any orthonormal basis set.

Define $|\psi_i\rangle = U|\psi_i\rangle$.

The $|\psi_i\rangle$'s also form an orthonormal basis set since unitary operators preserve inner products.

$$U = \sum_i |\psi_i\rangle \langle \psi_i|$$

$$U|\psi_i\rangle = \sum_j |\psi_j\rangle \langle \psi_j| |\psi_i\rangle = |\psi_i\rangle$$

Conversely, if $|\psi_i\rangle$ and $|\psi_j\rangle$ are any two orthonormal bases, then it is easily checked that $U$ defined by

$$U = \sum_i |\psi_i\rangle \langle \psi_i|$$

is unitary!

Proof: $U^*U = \left( \sum_i |\psi_i\rangle \langle \psi_i| \right)^+ \left( \sum_j |\psi_j\rangle \langle \psi_j| \right)$

$$= \sum_i \sum_j |\psi_i\rangle \langle \psi_i| |\psi_j\rangle \langle \psi_j|$$

$$= \sum_i |\psi_i\rangle \langle \psi_i| = I \rightarrow U \text{ is unitary.}$$

Completeness relation