OUTER PRODUCT p. 67

There is a useful way of representing linear operators which makes use of the inner product.

Take $|v\rangle$ in inner product space V
$|w\rangle$ " " " " W

Define $|w\rangle\langle v|$ to be the linear operator from $V \to W$ such that

$$\left(|w\rangle\langle v|\right)\underbrace{\left(|v'\rangle\right)}_{\in V} \equiv |w\rangle\langle v|v'\rangle = \underbrace{\langle v|v'\rangle}_{\in \mathbb{C}}\,\underbrace{|w\rangle}_{\in W}$$

Linear combination of outer products:

$$\left(\sum_i a_i |w_i\rangle\langle v_i|\right)\left(|v'\rangle\right) \triangleq \sum_i a_i \langle v_i|v'\rangle\, |w_i\rangle$$

Usefulness of outer product notation:

Consider an orthonormal basis set $|i\rangle$

$$|v\rangle = \sum_i^d v_i |i\rangle$$

$$\langle i|v\rangle = v_i$$

So $\left(\sum_i |i\rangle\langle i|\right)|v\rangle = \sum_i |i\rangle\langle i|v\rangle = \sum_i v_i |i\rangle = |v\rangle$

This is true for all $|v\rangle$'s in V!

$$\Rightarrow \boxed{\sum_i^d |i\rangle\langle i| = \mathbb{1}}$$

d is dimension of V

This is known as the Completness relation

problem: check it is trough for bases $[|0\rangle, |1\rangle]$ & $[|v_1\rangle, |v_2\rangle]$

First use of Completeness relation

Representation of an operator in the outer product notation

Suppose we have a linear operator $A : V \to W$

$|v_i\rangle$ is an orthonormal basis in V

$|w_j\rangle$ " " " " " W

Then $A = I_W A I_V$

So, $A = \sum_j |w_j\rangle\langle w_j| A \sum_i |v_i\rangle\langle v_i|$

$$A = \sum_{i,j} \langle w_j|A|v_i\rangle \, |w_j\rangle\langle v_i|$$

which is called the outer product representation of A.

Exercise 2.9 p. 68

Pauli operators and the outer product. $V = W = \mathbb{C}^2$

The Pauli matrices can be considered as operators with respect to an orthonormal basis $|0\rangle, |1\rangle$ for a two-dimensional Hilbert space.

Express each of the Pauli matrices in the outer product notation ($\sigma_x, \sigma_y, \sigma_z$)

$$\sigma_x = X = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \& \ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longleftarrow [\text{selected basis vectors}]$$

$$\sigma_x = \sum_{i,j=0}^{1} \overset{?}{\langle i|\sigma_x|j\rangle} |i\rangle\langle j|$$

$$\langle 0|\sigma_x|1\rangle = [1 \ \ 0]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1 \ \ 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

$$\langle 0|\sigma_x|0\rangle = [1 \ \ 0]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1 \ \ 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$\langle 1|\sigma_x|0\rangle = [0 \ \ 1]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [0 \ \ 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

$$\langle 1|\sigma_x|1\rangle = [0 \ \ 1]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [0 \ \ 1]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow \quad \sigma_x = +1 \ |0\rangle\langle 1| + 0 \ |0\rangle\langle 0| + 1 \ |1\rangle\langle 0| + 0 |1\rangle\langle 1|$$

$$\boxed{\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|}$$

The outer product representations of σ_y and σ_z can be derived the same way.

Note: The outer product representation would be different if we had selected another orthonormal base in $\mathbb{C}^2$

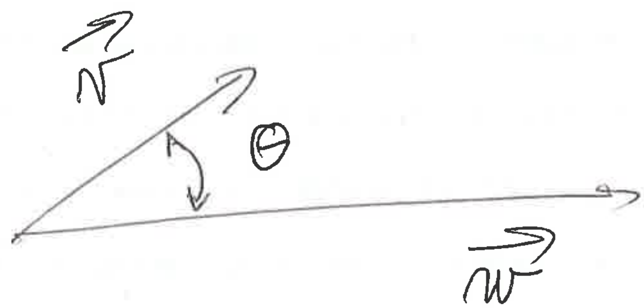
As an exercise, you could repeat the analysis above using the basis $|v_1\rangle, |v_2\rangle$ on page 4.

Review ~~Linear Operator~~ in $\mathbb{R}^3$

Before 19

$$|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\,\|\vec{w}\|$$ Cauchy-Schwartz inequality

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$$



$$-1 \leq \cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} \leq +1$$

$$\rightarrow \boxed{|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\,\|\vec{w}\|}$$

Second application of the completness relation

The CAUCHY-SCHWARTZ inequality

$$\boxed{\forall |v\rangle, |w\rangle \in V \text{ (dimension } d\text{)} \quad |\langle v|w\rangle|^2 \leq \langle v|v\rangle\langle w|w\rangle}$$

We use the Gram-Schmidt procedure to construct an orthonormal basis of vectors $|i\rangle$ with the first $|i\rangle$ as

$$|i\rangle = \frac{|w\rangle}{\| \, |w\rangle \|} = \frac{|w\rangle}{\sqrt{\langle w|w\rangle}}$$

Using $\sum_i |i\rangle\langle i| = \mathbb{1}$ (completness relation)

$$\langle v|v\rangle\langle w|w\rangle = \sum_{i=1}^{d} \langle v|i\rangle\langle i|v\rangle\langle w|w\rangle$$

Dropping some non-negative terms (keeping only $i=1$ term)

$$\langle v|v\rangle\langle w|w\rangle \geq \underbrace{\left[\frac{\langle v|w\rangle\langle w|v\rangle}{\left(\sqrt{\langle w|w\rangle}\right)^2}\right]\langle w|w\rangle}_{\langle v|w\rangle\langle w|v\rangle}$$

Hence $\langle v|w\rangle\langle v|w\rangle^* \leq \langle v|v\rangle\langle w|w\rangle$

or $\boxed{|\langle v|w\rangle|^2 \leq \langle v|v\rangle\langle w|w\rangle}$

Equality occurs if $|v\rangle = z\,|w\rangle$, i.e., iif the 2 vectors are linearly related (z is a scalar

Section 2.1.5 Eigenvectors and Eigenvalues

Definition (a) Eigenvector of linear operator A is a non-zero vector $|v\rangle$ such that

$$A|v\rangle = \lambda |v\rangle$$

where v is a complex number known as the corresponding eigenvalue.

(b) Eigenvalues? are solutions of

$$\underbrace{\det |A - \lambda \mathbb{1}|}_{\text{polynomial in } \lambda} = 0$$

The eigenspace corresponding to an eigenvalue v is the set of vectors with eigenvalue v. It is a vector subspace of the vector space on which A acts.

Diagonal representation of A also known as orthonormal Decomposition of A

$$\boxed{A = \sum_i \lambda_i |i\rangle\langle i|}$$

where $|i\rangle$ vectors form an orthonormal set of eigenvectors of A.

An operator is diagonalizable if it has a diagonal representation. Soon, we will see the necessary and sufficient conditions for an operator to be diagonalizable

Example

20b

Intro to Quantum Computing : ECES 622 - Winter 2007

Midterm Exam: Tuesday February 13, 2007

Problem I: Consider the following matrix A

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{1}$$

- (1) Calculate the eigenvalues and corresponding eigenvectors of this matrix. Normalize the eigenvectors.
- (2) What are the angles θ and ϕ in the general expression of the qubit on the Bloch sphere, i.e.,

$$|\xi_n^+> = cos\frac{\theta}{2}|0> + sin\frac{\theta}{2}e^{i\phi}|1>, \tag{2}$$

associated to the two eigenvectors of the matrix A?

- (3) Give the outer product representation of A using the basis formed by the two kets $|0>$ and $|1>$, i.e, what are the coefficients (α, β, γ, and δ) in the decomposition:

$$A = \alpha|0><0| + \beta|0><1| + \gamma|1><0| + \delta|1><1|? \tag{3}$$

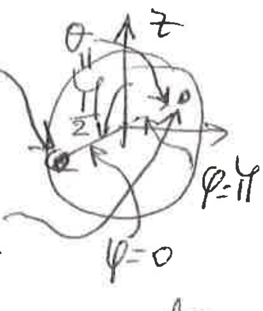
- (4) Give the expression of a *non-diagonal* matrix which commutes with the matrix A. Explain how you obtained that matrix B.

(1) $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)^2 - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0 \rightarrow \lambda = \frac{2 \pm \sqrt{4+12}}{2} = 3, -1$

$\lambda = 3 \rightarrow \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle_x$

(2)



$\lambda = -1 \rightarrow \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle_x$

(3) $A = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

α $|0\rangle\langle 0|$ β $|0\rangle\langle 1|$ γ $|1\rangle\langle 0|$ δ $|1\rangle\langle 1|$

(4) $X = B$ has eigenvectors $|+\rangle_x$ $|-\rangle_x$ → Commute with A.

Example $\sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has obviously eigenvalues 1 and -1

Furthermore $\sigma_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = +1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\sigma_z |0\rangle = +1 |0\rangle$

$\sigma_z \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\sigma_z |1\rangle = -1 |1\rangle$

Therefore the diagonal representation of σ_z is
(orthonormal decomposition)

$$\sigma_z = (+1)\, |0\rangle\langle 0| + (-1)\, |1\rangle\langle 1|$$

$$\boxed{\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|}$$

Backwards

$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (Kronecker product)

$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$Z = \sigma_z$ is indeed $= |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\| $$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Section 2.1.6 Adjoint and Hermitian Operators

If A is a linear operator on a Hilbert space V there exists a unique operator $A^\dagger$ ($\dagger$ is pronounced dagger) on V such that

$\forall\ |v\rangle, |w\rangle \in V,$
$$(|v\rangle, A|w\rangle) = (A^\dagger|v\rangle, |w\rangle)$$

$A^\dagger$ is called ADJOINT or HERMITIAN CONJUGATE of A.

If A, B are linear operators on V, $A^\dagger, B^\dagger$ their respective adjoints, then $(AB)^\dagger = B^\dagger A^\dagger$

Proof:
$$\boxed{(|v\rangle, AB|w\rangle)} = \boxed{((AB)^\dagger|v\rangle, |w\rangle)}$$
$$\|$$
$$(|v\rangle, A[B|w\rangle]) = (A^\dagger|v\rangle, B|w\rangle)$$
$$= ((B^\dagger A^\dagger)|v\rangle, |w\rangle)$$

So, indeed $\boxed{(AB)^\dagger = B^\dagger A^\dagger}$

By convention, $|v\rangle^\dagger \equiv \langle v|$ and $|v\rangle \equiv \langle v|^\dagger$

So $(A|v\rangle)^\dagger = \langle v| A^\dagger$

If $A^\dagger = A$ A is Hermitian

Properties of Adjoint

① $(A^\dagger)^\dagger = A$

② $\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* A_i^\dagger$ adjoint operator is antilinear

③ $(|v\rangle\langle v|)^\dagger = |v\rangle\langle v|$ $\Rightarrow$ $|v\rangle\langle v|$ is Hermitian for any vector $|v\rangle$

Matrix Representation of Adjoint

$$A^\dagger \equiv (A^*)^T$$

$*$ stands for complex conjugation

T " " transpose

Example:

$$A = \begin{bmatrix} 1+3i & 2i \\ 1+i & 1-4i \end{bmatrix}$$

$$A^\dagger = \begin{bmatrix} 1-3i & -2i \\ 1-i & 1+4i \end{bmatrix}^T = \begin{bmatrix} 1-3i & 1-i \\ -2i & 1+4i \end{bmatrix}$$

An operator whose adjoint is A itself

$A^\dagger = A$ $\Downarrow$ diagonal elements are real

$\boxed{\text{is said to be SELF-ADJOINT or HERMITIAN}}$

$(A^+)^+ = A.$ $\boxed{\text{Proof}}$ Call $B = A^+$ (230)

$$(|v\rangle, B^+|w\rangle)$$
$$\|$$
$$(B|v\rangle, |w\rangle)$$
$$\|$$
$$(A^+|v\rangle, |w\rangle)$$
$$\|$$
$$(|v\rangle, A|w\rangle)$$

Equal $\to B^+ = A$

or

$(A^+)^+ = A$

$$\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* A_i^\dagger$$

$$(A+B)^\dagger = A^\dagger + B^\dagger$$

First

$$\left(|v\rangle, (A+B)|w\rangle\right) = \left((A+B)^\dagger |v\rangle, |w\rangle\right)$$

$$\left(|v\rangle, (A+B)|w\rangle\right) \overset{(1)}{=} \left((A+B)^\dagger |v\rangle, |w\rangle\right)$$

$$\overset{(2)}{=}$$

$$\left(|v\rangle, A|w\rangle + B|w\rangle\right) = \left(|v\rangle, A|w\rangle\right) + \left(|v\rangle, B|w\rangle\right)$$

$$= \left(A^\dagger |v\rangle, |w\rangle\right) + \left(B^\dagger |v\rangle, |w\rangle\right)$$

$$= \left((A^\dagger + B^\dagger)|v\rangle, |w\rangle\right)$$

So

$(A+B)^\dagger = A^\dagger + B^\dagger$ <u>indeed</u>

Second $(\alpha A)^\dagger = \alpha^* A^\dagger$

$$\left(|v\rangle, \alpha A|w\rangle\right)^{(1)}$$

$$\overset{(2)}{=}$$

$$\alpha\left(|v\rangle, A|w\rangle\right)$$

$$=$$

$$\alpha\left(A^\dagger|v\rangle, |w\rangle\right)$$

$$=$$

$$\alpha\left(|w\rangle, A^\dagger|v\rangle\right)^*$$

$$\to (\alpha^*)^*\left(|w\rangle, A^\dagger|v\rangle\right)^*$$

$$=$$

$$\left[\alpha^*\left(|w\rangle, A^\dagger|v\rangle\right)\right]^*$$

$$=$$

$$\left[\left(|w\rangle, \alpha^* A^\dagger|v\rangle\right)\right]^*$$

$$=$$

$$\left(\alpha^* A^\dagger|v\rangle, |w\rangle\right)$$

$$=$$

$$\to \left((\alpha A)^\dagger|v\rangle, |w\rangle\right)$$

Property ③ p.23

$$\boxed{\Big(\underbrace{|w\rangle\langle v|}_{A}\Big)^\dagger \overset{?}{=} |v\rangle\langle w|}$$

$$\big(|v\rangle,\ A|w\rangle\big) = \big(|v\rangle,\ |w\rangle\langle v|w\rangle\big)$$

$$= \langle v|w\rangle\,\big(|v\rangle, |w\rangle\big)$$

$$= \Big(\overbrace{|v\rangle\langle w|}^{A^\dagger}\underbrace{\langle w|v\rangle}_{\in \mathbb{C}},\ |w\rangle\Big)$$

$$= \langle w|v\rangle^*\,\big(|v\rangle, |w\rangle\big)$$

$$= \langle v|w\rangle\,\big(|v\rangle, |w\rangle\big)$$

equal

Important Class of Hermitian operators

PROJECTORS

Take W a k-dimensional subspace of d-dimensional vector space V

Using the Gram-Schmidt procedure, it is possible to construct an orthonormal basis $|1\rangle \dots |d\rangle$ for V (of dimension d) such that $|1\rangle, \dots, |k\rangle$ is an orthonormal basis for W.

By definition,

$$\boxed{P = \sum_{i=1}^{k} |i\rangle\langle i|}$$

is the projector onto the subspace W.

$|v\rangle\langle v|$ is Hermitian for any vector $|v\rangle$, hence P is Hermitian, i.e., $P^{\dagger} = P$ (PROVE IT!)

By definition, the orthogonal complement of P is the operator $Q = I - P$. Q is a projector onto the vector space spanned by the vectors $|k+1\rangle, \dots, |d\rangle$

Completeness relation:

$$\underbrace{\sum_{i=1}^{d} |i\rangle\langle i|}_{\mathbb{1}} = \underbrace{\sum_{i=1}^{k} |i\rangle\langle i|}_{P} + \underbrace{\sum_{i=k+1}^{d} |i\rangle\langle i|}_{Q}$$

- An operator is normal if $A^{\dagger}A = AA^{\dagger}$

Clearly, any Hermitian operator is also normal.

Important Theorem: An Operator is a normal operator iif it is diagonalizable

This is called the SPECTRAL DECOMPOSITION THM P. 72

- A matrix is unitary if $U^{\dagger}U = \mathbb{I}$

Similarly, an operator is unitary if $U^{\dagger}U = \mathbb{I}$

$$\text{Clearly, } U^{\dagger}U = \mathbb{I} = UU^{\dagger}$$

A unitary operator U is normal and therefore also has a spectral decomposition.

- What's so important about unitary operators?

As we will see later, they play a key role in quantum computing as any action on qubit will be using unitary operators.

These operators preserve inner products between vectors.

$\forall |v\rangle, |w\rangle$, If U is a unitary operator

$$(U|v\rangle, U|w\rangle) = \langle v|U^{\dagger}U|w\rangle = \langle v|\mathbb{I}|w\rangle = \langle v|w\rangle$$

$= (|v\rangle, U^{\dagger}U|w\rangle)$ $(|v\rangle, |w\rangle)$

Outer Product Representation of any unitary operator U

The previous result leads to the following elegant outer product representation of any operator U (unitary)

Take $|v_i\rangle$ any orthonormal basis set

Define $|w_i\rangle \equiv U|v_i\rangle$

The $|w_i\rangle$'s also form an orthonormal basis set since unitary operators preserve inner products

$$U = \sum_j |w_j\rangle\langle v_j|$$

$$U|v_i\rangle = \sum_j |w_j\rangle \underbrace{\langle v_j|v_i\rangle}_{\delta_{ij}} = |w_i\rangle$$

Conversely, if $|v_i\rangle$ and $|w_i\rangle$ are any two orthonormal bases, then it is easily checked that U defined by

$$U \equiv \sum_i |w_i\rangle\langle v_i|$$

is unitary!

Proof:
$$U^\dagger U = \left(\sum_i |w_i\rangle\langle v_i|\right)^\dagger \left(\sum_j |w_j\rangle\langle v_j|\right)$$

$$= \sum_i\sum_j |v_i\rangle \underbrace{\langle w_i|w_j\rangle}_{\delta_{ij}} \langle v_j|$$

$$= \sum_i |v_i\rangle\langle v_i| = \mathbb{1} \longrightarrow U \text{ is unitary.}$$

Completness relation