Take \( x^2 = -3 \)

It does not have real solutions, i.e., it is not satisfied by any real number.

We introduce an imaginary number \( j \) such that \( j^2 = -1 \)

\[ j = \sqrt{-1} \]

\[ j^2 = -1 \]

\[ j^3 = j^2 j = -j \]

\[ j^4 = j^2 j^2 = +1, \ldots \]

The sum of a real number and imaginary number (product of \( j \) by a real number) is called a complex number, of the form,

\[ a+jb \]

with \( a \) and \( b \) real. We use the notation

\[ A = a+jb \]

\[ \text{Re}\, A = a \]

\[ \text{Im}\, A = b \]

\[ A \text{ has 2 components. We can represent complex numbers graphically using Cartesian coordinates.} \]

\[ A \text{ complex plane} \]

\[ A \text{ Argand diagram} \]

\[ A = a+jb \]
Two complex numbers are equal if their real parts are equal and their imaginary parts are equal.

\[ \text{Sum} \quad (a + jb) + (c + jd) = \left( \frac{a + c}{\text{sum of real parts}} \right) + j \left( \frac{b + d}{\text{sum of imaginary parts}} \right) \]

\[ \text{Difference} \quad (a + jb) - (c + jd) = \left( \frac{a - c}{\text{diff of real parts}} \right) + j \left( \frac{b - d}{\text{diff of imaginary parts}} \right) \]

**Graphically**

\[ M = 3 + j \]
\[ N = 2 - 2j \]
\[ M + N = 5 - j \]

**Product**

\[ (a + jb)(c + jd) = (ac - bd) + j(bc + ad) \]

**Example**

\[ (3 + j4)(4 - j2) = 12 - j6 + j16 - j^2 8 \]
\[ = 12 + j10 + 8 \]
\[ = 20 + j10 \]
First, we define the conjugate of a complex number.

**Notation** \( A^* = a - jb \) if \( A = a + jb \).

The conjugate of a complex number is merely taken by changing the sign of the imaginary part.

**Example** \( A = 5 + j3 \) \( \Rightarrow \) \( A^* = 5 - j3 \)

**Properties** \( (A^*)^* = A \)

**Quotient of 2 complex numbers**

\[
\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}
\]

**Example**

\[
\frac{3 + j4}{1 - j2} = \frac{(3 + j4)(1 - j2)}{1^2 + 4} = \frac{12 + j6 + j16 - 8}{20} = \frac{4 + j22}{20} = 0.2 + j1.1
\]

**Cauchy's identity**

**Taylor expansions**

\[
\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots
\]

\[
\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots
\]
\[ \cos \theta + j \sin \theta = 1 + j \theta - \frac{\theta^2}{2!} - j \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j \frac{\theta^5}{5!} \]

\[ R = 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \frac{r^5}{5!} + \ldots \]

\[ e^{j\theta} = 1 + j \theta + \frac{\theta^2}{2!} + j \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j \frac{\theta^5}{5!} + \ldots \]

\[ = 1 + j \theta - \frac{\theta^2}{2!} - j \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j \frac{\theta^5}{5!} + \ldots \]

\[ A = e^{j\theta} = \cos \theta + j \sin \theta \]

\[ A = \cos \theta + j \sin \theta = \cos(-\theta) + j \sin(-\theta) = e^{-j\theta} \]

\[ (1) + (2) \Rightarrow \cos \theta = \frac{1}{2} (e^{j\theta} + e^{-j\theta}) \]

\[ (1) - (2) \Rightarrow \sin \theta = \frac{1}{2j} (e^{j\theta} - e^{-j\theta}) \]

---

For a complex number, define:

\[ \cos z = \frac{1}{2} (e^{jz} + e^{-jz}) \]

\[ \sin z = \frac{1}{2j} (e^{jz} - e^{-jz}) \]
The exponential form of complex numbers

\[ e^{\theta} = \cos \theta + j \sin \theta \]

\[ C \ e^{\theta} = \sqrt{C \cos \theta + j C \sin \theta} = a + j b \]

Positive real \quad Real \quad Imaginary

\[ a = C \cos \theta \]
\[ b = C \sin \theta \]
\[ a^2 + b^2 = C^2 \Rightarrow C = \sqrt{a^2 + b^2} \]

\[ \frac{b}{a} = \tan \theta \Rightarrow \theta = \tan^{-1} \left( \frac{b}{a} \right) \]

So we can find \( a \) and \( \theta \) from a knowledge of \( a \) and \( b \).

Example

\[ A = 4 + j 2 \]
\[ C = \sqrt{16 + 4} = 4.47 \]
\[ \theta = \tan^{-1} \left( \frac{2}{4} \right) = 26.6^0 \]

So in cartesian form

\[ A = \frac{C \cos \theta + j C \sin \theta}{4.47 \cos 26.6^0 + j 4.47 \sin 26.6^0} \]

\[ A = 4 + j 2 \]

A complex number represented in this manner is said to be in exponential form.
\[ A = C \cdot e^{j\theta} \] argument or angle

magnitude or amplitude

\[ \theta \] is expressed in degrees

\[ \theta = \tan^{-1} \left( \frac{b}{a} \right), \tan^{-1} \] is a multivalued function.

which one to choose?

Example

\[ V = y - j3 \]

\[ C = \sqrt{y^2 + 3^2} = 5 \]

\[ \theta = \tan^{-1} \left( -\frac{3}{y} \right) \]

\[ a = 4 = 5 \cos \theta \rightarrow \theta = -36.9^\circ, 323.1^\circ - 36.9^\circ \]

\[ b = -3 = 5 \sin \theta \]

choose the lowest one \( \theta = -36.9^\circ \)

\[ \theta = \tan^{-1} \left( -\frac{3}{y} \right) \]

This value is not acceptable because \( \cos \theta \) would be negative and \( \sin \theta \) would be positive.

Is there a simpler method to select the correct \( \theta \)?

Yes, using the graphical representation.

\[ \tan \theta = \tan (\theta + \phi) \]

\[ \frac{\sin \theta}{\cos \theta} = \frac{\sin (\theta + \phi)}{\cos (\theta + \phi)} = \frac{\sin (\theta - \phi)}{\cos (\theta - \phi)} = -\frac{\sin (-\theta)}{-\cos \phi} = \tan \theta \]
Take \( V = 4 - j^3 \)

\[ \theta = 323.1^\circ \]

\[ V = 4 - j^3 = 5 \, \text{e}^{-j36.9^\circ} \]

\( \text{Cartesian form} \quad \text{Exponential form} \)
Example

\[ I = -5 + j2 \]

i.e., real part is negative \( \rightarrow \) 3rd quadrant

\[ I = -(5 - j2) \]

\[ 5 - j2 = C e^{j\theta} \quad C = \sqrt{\frac{29}{5}} = 5.39 \]

\[ \theta = \tan^{-1} \left( \frac{-2}{5} \right) = -21.8^\circ \]

\[ \rightarrow I = -C e^{j\theta} = -5.39 e^{-j21.8^\circ} \]

\[ -1 = e^{j\pi} = e^{j180^\circ} \]

\[ \Rightarrow I = 5.39 e^{-j158.2^\circ} \quad \text{or} \quad I = 5.39 e^{-j201.8^\circ} \]

\[ A = C e^{j\theta} \quad B = D e^{j\phi} \]

\[ A = B \quad \text{if} \quad C = D \quad \text{and} \quad \theta = \phi \pm m(360^\circ) \]

\[ m = 0, 1, 2, 3 \cdots \]

The polar form

\[ A = C e^{j\theta} \quad \text{is sometimes written more concisely as} \quad A = C / \theta \]

\[ A = -2 + j5 = 5.39 e^{j111.8^\circ} = 5.39 /111.8^\circ \]

are 3 different ways to represent a complex number.
Multiplication of complex numbers using the exponential and polar forms.

\[ A = 5 \angle 53.1^\circ \quad B = 15 \angle -36.9^\circ \]

\[ \Rightarrow A = 5 \, e^{j 53.1^\circ} \quad B = 15 \, e^{-j 36.9^\circ} \]

\[ \Rightarrow AB = 5 \times 15 \, e^{j (53.1^\circ - 36.9^\circ)} = 75 \, e^{j 16.2^\circ} = 75 \angle 16.2^\circ \]

\[ \frac{A}{B} = \frac{5 \, e^{j 53.1^\circ}}{15} \cdot \frac{e^{-j 36.9^\circ}}{e^j 36.9^\circ} = 0.333 \, e^{j 0^\circ} \]

So \[ \frac{A}{B} = 0.333 \, e^{j 0^\circ} \]

Summary:
As for addition and subtraction, it is easier to work with rectangular form of complex numbers.
For multiplication and division, it is better to work with exponential form.

Different forms to represent a complex number:
\[ A = a + j b = \text{Re} \, A + j \, \text{Im} \, A = Ce^{j \theta} \]
\[ = \sqrt{a^2 + b^2} \angle \tan^{-1}(b/a) = \sqrt{a^2 + b^2} / \tan^{-1}(b/a) \]
Express each of the following complex numbers in exponential form, using an angle lying in the range 

$-180^\circ < \theta < 180^\circ$.

$M = -18.5 - j26.1$

$M' = 18.5 + j26.1 = \sqrt{(18.5)^2 + (26.1)^2} \ e^{j\theta}$

$\theta = \tan^{-1} \left( \frac{26.1}{18.5} \right) = 54.7^\circ$

$M = \sqrt{(18.5)^2 + (26.1)^2} \ e^{j\theta} = 32 \ e^{j(54.7 + 180^\circ)}$

$= 32 \ e^{j(234.7^\circ)}$

$\Rightarrow \quad M = 32 \ e^{j125.3^\circ}$
Practice 45.3

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

\[ e^{-j\theta} = \cos \theta - j \sin \theta \]

2 rad = 360°
1 rad = 180°

(c) \[ e^{j(180°)} = \cos \left( \frac{180°}{\pi} \right) + j \sin \left( \frac{180°}{\pi} \right) = 0.54 + j0.84 \]

(b) \[ e^{j \left( \frac{-180°}{\pi} \right)} = e^{-j \left( \frac{180°}{\pi} \right)} = e^{-j \cdot \text{(previous)} \text{ (answer)}} = 1.464 - j2.29 \]

(a) \[ \cos(-j\theta) = \frac{1}{2} \left( e^{-j\theta} + e^{j\theta} \right) = 1.543 \]

\[ \sin(-j\theta) = \frac{-1}{2j} \left( e^{j\theta} - e^{-j\theta} \right) = -j(1.175) \]
Express the following complex number in rectangular form.

\[ 61.2 \times e^{j111.1} = 61.2 \cos 111.1 - j \times 61.2 \sin 111.1 = -22.0 - j57.1 \]

Express the following complex number in polar form.

\[ Z = (\overline{z} - 1) / (0.3 \times 41^\circ) \]

\[ = \left[ (\overline{z} - (1 \cos(-41^\circ) + j \sin(-41^\circ))) \right] / (0.3 \times e^{j41^\circ}) \]

\[ = \left[ (\overline{z} - \cos 41^\circ) - j \sin 41^\circ \right] \times \frac{1}{0.3} \times e^{-j41^\circ} \]

\[ = \sqrt{(\overline{z} - \cos 41^\circ)^2 + \sin 41^\circ} \times e^{j \tan^{-1}\left(\frac{-\sin 41^\circ}{2 - \cos 41^\circ}\right) - j41^\circ} \]

\[ = 4.69179 e^{-j13.2183} \]

\[ Z = 4.69179 / -13.2183 \]