Schaum's Outline Series
Theory and Problems of Modern Physics.
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Chapter 21

The Stern–Gerlach Experiment and Electron Spin

21.1 THE STERN–GERLACH EXPERIMENT

In the *Stern–Gerlach experiment*, performed in 1921, a beam of silver atoms having zero total orbital angular momentum passes through an *inhomogeneous* magnetic field and strikes a photographic plate, as shown in Fig. 21-1. Any deflection of the beam when the magnetic field is turned on is measured on the photographic plate.

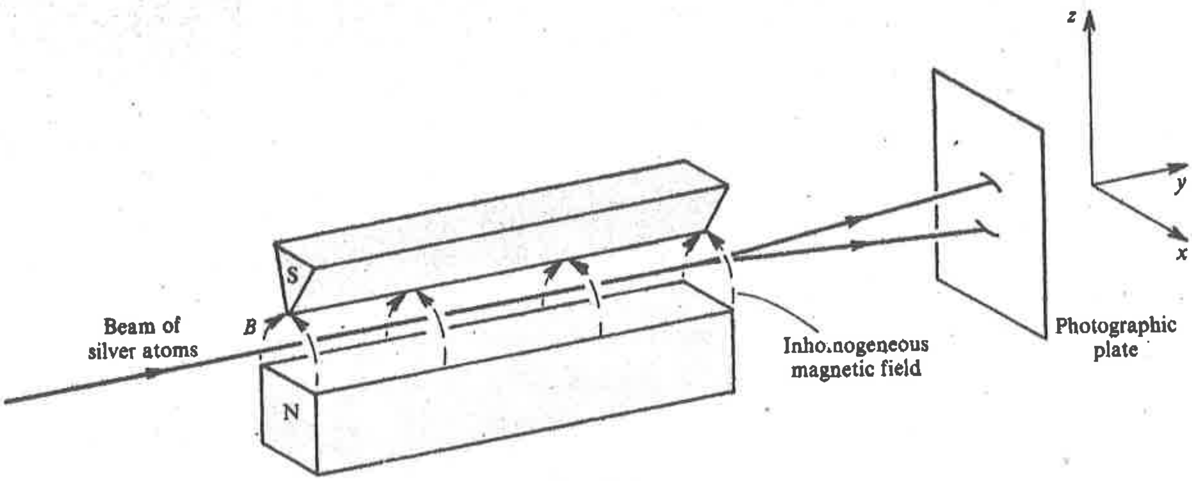

Fig. 21-1

The purpose of the *inhomogeneous* magnetic field is to produce a deflecting force on any magnetic moments that are present in the beam. If a homogeneous magnetic field were used, each magnetic moment would experience only a torque and no deflecting force. In an inhomogeneous magnetic field, however, a net deflecting force will be exerted on each magnetic moment μ_s. For the situation of Fig. 21-1,

$$F_z = \mu_s \cos\theta \frac{dB}{dz} \tag{21.1}$$

where θ is the angle between μ_s and **B**, and dB/dz is the gradient of the inhomogeneous field (see Problem 21.1).

In the experiment it is found that when the beam strikes the photographic plate it has split into two distinct parts, with equal numbers of atoms deflected above and below the point where the beam strikes when there is no magnetic field. Because the atoms have zero total orbital angular momentum and therefore zero magnetic moment due to orbital electron motion, the magnetic interaction that produced the deflections must come from another type of magnetic moment.

21.2 ELECTRON SPIN

In 1925, S. A. Goudsmit and G. E. Uhlenbeck suggested that an electron possesses an *intrinsic* angular momentum called its *spin*. The extra magnetic moment μ_s associated with the intrinsic spin angular momentum **S** of the electron accounts for the deflection of the beam observed in the Stern–Gerlach experiment.

Similar to the orbital angular momentum, the electron's intrinsic angular momentum and associated magnetic moment are quantized both in magnitude and direction. The two equally spaced lines observed in the Stern–Gerlach experiment show that the intrinsic angular momentum can assume only two orientations with respect to the direction of the impressed magnetic field. In Section 20.6 it was shown that for orbital motion specified by the quantum number l, the component of the orbital magnetic moment along the magnetic field can have $2l + 1$ discrete values. Similarly, if the quantum number for the spin angular momentum is specified by s, we have, since there are only two orientations possible, $2 = 2s + 1$, giving the unique value $s = 1/2$. The magnitude of the spin angular momentum **S** is then

$$|\mathbf{S}| = \sqrt{s(s+1)}\,\hbar = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)}\,\hbar = \frac{\sqrt{3}}{2}\hbar \tag{21.2}$$

The component S_z along the z-direction is

$$S_z = m_s\hbar \qquad m_s = s,\, s-1 = \tfrac{1}{2},\, -\tfrac{1}{2} \tag{21.3}$$

The two orientations of **S** are commonly referred to as "spin up" ($m_s = +\frac{1}{2}$) and "spin down" ($m_s = -\frac{1}{2}$) (although the spin can never point in the positive or negative z-direction).

It is also found that the electron's intrinsic magnetic moment μ_s and intrinsic angular momentum **S** are proportional to each other; their relationship can be written as

$$\mu_s = -g_s \frac{e}{2m}\mathbf{S} \tag{21.4}$$

The dimensionless quantity g_s is called the *gyromagnetic ratio*; for the electron, it has the value 2.002 (we shall use $g_s = 2.0$ in the problems). A comparison of (*21.4*) with (*20.2*) gives

$$g_s = \frac{|\mu_s|/|\mathbf{S}|}{|\mu|/|\mathbf{L}|}$$

Thus the ratio of magnetic moment to angular momentum is about twice as great for electron spin as it is for the electron's orbital motion.

The unique value 1/2 for the spin quantum number is a characteristic as basic to the electron as its unique charge and mass. The properties of electron spin were first explained by Dirac around 1928, by combining the principles of wave mechanics with the theory of relativity. It should be noted that particles other than electrons, e.g. protons and neutrons, also possess an intrinsic angular momentum.

Solved Problems

21.1. Derive (*21.1*).

The potential energy of an electron in a magnetic field is [cf. (*20.4*)]

$$E_B = -\mu_s \cdot \mathbf{B} = -\mu_{sx}B_x - \mu_{sy}B_y - \mu_{sz}B_z$$

For the field of Fig. 21-1, $B_y \equiv 0$, and B_x and B_z depend only on x and z. Therefore,

$$F_x = -\frac{\partial E_B}{\partial x} = \mu_{sx}\frac{\partial B_x}{\partial x} + \mu_{sz}\frac{\partial B_z}{\partial x}$$

$$F_y = -\frac{\partial E_B}{\partial y} = 0$$

$$F_z = -\frac{\partial E_B}{\partial z} = \mu_{sx}\frac{\partial B_x}{\partial z} + \mu_{sz}\frac{\partial B_z}{\partial z}$$

But, along the beam axis, $\partial B_z/\partial x = 0$ (by symmetry) and $B_x = \partial B_x/\partial z = 0$ (by antisymmetry); also, $\partial B_x/\partial x$ will be very small. Consequently,

$$F_x \approx 0 \qquad F_y = 0 \qquad F_z = \mu_{sz}\frac{dB}{dz} = \mu_s \cos\theta \frac{dB}{dz}$$

21.2. Determine the maximum separation of a beam of hydrogen atoms that moves a distance of 20 cm with a speed of 2×10^5 m/s perpendicular to a magnetic field whose gradient is 2×10^2 T/m. Neglect the magnetic moment of the proton (see Problem 26.9).

In the ground state, hydrogen atoms have zero orbital angular momentum. From Problem 21.1, the force on a hydrogen atom is

$$F_z = \mu_{sz}\frac{dB}{dz}$$

By (*21.3*) and (*21.4*), with $g_s = 2$, $\mu_{sz} = -(e/m)m_s\hbar$, so that

$$|F_z| = \frac{e\hbar}{m}|m_s|\frac{dB}{dz} = \frac{e\hbar}{2m}\frac{dB}{dz} = \left(9.27 \times 10^{-24}\,\frac{\text{J}}{\text{T}}\right)\left(2 \times 10^2\,\frac{\text{T}}{\text{m}}\right) = 1.85 \times 10^{-21}\text{ N}$$

Using the constant-acceleration formulas $\Delta z = \frac{1}{2}a_z t^2$ and $\Delta y = vt$ (see Fig. 21-1 for the coordinates), we obtain

$$\Delta z = \frac{1}{2}a_z t^2 = \frac{1}{2}\left(\frac{F_z}{m_H}\right)\left(\frac{\Delta y}{v}\right)^2$$

The mass of hydrogen is 1.67×10^{-27} kg, so

$$\Delta z = \frac{1}{2}\left(\frac{1.85 \times 10^{-21}\text{ N}}{1.67 \times 10^{-27}\text{ kg}}\right)\left(\frac{0.20\text{ m}}{2 \times 10^5\text{ m/s}}\right)^2 = 5.54 \times 10^{-7}\text{ m}$$

Since this is the displacement up or down, the total separation is $2\,\Delta z = 1.11 \times 10^{-6}$ m.

21.3. Determine the energy difference between the electrons that are "aligned" and "anti-aligned" with a uniform magnetic field of 0.8 T when a beam of free electrons moves perpendicular to the field.

From Problem 21.1 with $B_x = B_y = 0$

$$E_B = -B\mu_{sz} = -B\left(-\frac{e\hbar}{m}\right)m_s$$

Hence,

$$\Delta E_B = B\frac{e\hbar}{m}\,\Delta m_s = (0.8\text{ T})\left(2 \times 5.79 \times 10^{-5}\,\frac{\text{eV}}{\text{T}}\right)\left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right] = 9.26 \times 10^{-5}\text{ eV}$$

21.4. The 21 cm line is used in radioastronomy to map the galaxy. The line arises from the emission of a photon when the electron in a galactic hydrogen atom "flips" its spin from being aligned to being anti-aligned with the spin of the proton in the hydrogen atom. What is the magnetic field the electron experiences?

$$\Delta E = \frac{hc}{\lambda} = \frac{12.4 \times 10^3\text{ eV}\cdot\text{Å}}{21 \times 10^8\text{ Å}} = 5.9 \times 10^{-6}\text{ eV}$$

From Problem 21.3 we have

$$\Delta E_B = B\frac{e\hbar}{m}\,\Delta m_s$$

$$5.9 \times 10^{-6}\text{ eV} = B\left[2 \times 5.79 \times 10^{-5}\,\frac{\text{eV}}{\text{T}}\right]\left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right]$$

$$B = 0.0510\text{ T}$$