Chapter 9

Electron Spin

*These sections can be omitted without loss of continuity.

9.1 Introduction

In Chapter 8 we saw how the Schrödinger equation can explain many properties of the hydrogen atom. Our next logical step would be to describe how Schrödinger's theory — unlike the Bohr model — also gives an excellent account of all higher, multielectron atoms. Before we can do this, however, we need to introduce another important property of the electron, its *spin angular momentum*, or *spin*. In this chapter we describe the electron's spin and several of its experimental consequences. Then in Chapter 10 we will return to the Schrödinger equation and use it to explain the properties of multielectron atoms.

In Section 9.2 we state the observed facts about the electron's spin angular momentum and its quantized values. Most of the more obvious manifestations of the electron's spin concern the magnetic effects associated with a spinning charged particle. Therefore, in Sections 9.3 to 9.5 we describe the magnetic properties of orbiting and spinning charged particles. Then in Sections 9.6 and 9.7 we describe several important experimental consequences of the electron's spin magnetic moment. These effects, all important in their own right, are historically important because they gave evidence for the existence of spin. Finally, in Section 9.8 we discuss an important technological application of spin: magnetic resonance imaging, a powerful medical diagnostic tool.

9.2 Spin Angular Momentum

As the earth orbits around the sun its total angular momentum $\mathbf{J}$ is the sum of two terms,

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{9.1}$$

Here the first term, **L**, is **r** × **p**, where **r** is the position vector of the earth relative to the sun and **p** is the earth's linear momentum. Because this term arises from the earth's yearly orbital motion, it is called the *orbital angular momentum*. The second term, **S**, is $I\boldsymbol{\omega}$, where I is the earth's moment of inertia and $\boldsymbol{\omega}$ is the angular velocity of its daily spinning on its own axis; this second term, **S**, is called the earth's *spin*. In a similar way, the angular momentum of an electron is found to be the sum of two terms with the same form (9.1). The first term, **L**, is the orbital angular momentum, and this is the angular momentum discussed in Chapter 8, with quantized magnitude $\sqrt{l(l+1)}\hbar$ and components $m\hbar$. The second term, **S**, is called the electron's **spin**. For most purposes, one can visualize the electron's spin as analogous to the earth's spinning motion as it rotates on its own axis.*

We saw in Chapter 8 that the magnitude of **L** is quantized, with allowed values

$$L = \sqrt{l(l+1)}\hbar$$

The magnitude of the spin **S** is found to be given by a similar formula,

$$S = \sqrt{s(s+1)}\hbar \tag{9.2}$$

Here the spin quantum number s determines the magnitude of the spin **S** in just the same way that l determines the magnitude of **L**. There is, however, an important difference: As we saw in Chapter 8, l can be any integer:

$$l = 0, 1, 2, 3, \cdots$$

Experiment shows that, for an electron, s always has a fixed value, which is not an integer, namely

$$s = \tfrac{1}{2} \tag{9.3}$$

According to (9.2), this means that the electron's spin **S** always has the same magnitude,

$$S = \sqrt{s(s+1)}\hbar = \sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)}\hbar = \frac{\sqrt{3}}{2}\hbar \tag{9.4}$$

and for this reason, the spin is sometimes described as the *intrinsic angular momentum* of the electron. Because the quantum number s is $\frac{1}{2}$, one often refers to the electron as having "spin half."

The possible values of the z component (or any other component) of the orbital angular momentum **L** have the form

$$L_z = m\hbar$$

where m runs in integer steps from l to $-l$:

$$m = l, l-1, \cdots, -l$$

*We should emphasize that this analogy is not exact. For example, if the electron were, like the earth, a spinning ball of matter, its spin angular momentum would be characterized by a quantum number, l, that could take on any integer value, $l = 0, 1, 2, \cdots$. As we will see shortly, the spin quantum number is *fixed* and *noninteger*.

It is found that a corresponding result holds for spin. The possible values of S_z are

$$S_z = m_s \hbar$$

where m_s is a quantum number that runs in integer steps from s to $-s$. However, since $s = \frac{1}{2}$, this gives only two possible values:

$$m_s = \tfrac{1}{2} \quad \text{or} \quad -\tfrac{1}{2}$$

and hence

$$S_z = \tfrac{1}{2}\hbar \quad \text{or} \quad -\tfrac{1}{2}\hbar \tag{9.5}$$

We often describe these two possibilities by saying the electron's spin can be "up" or "down," and represent these two states by arrows, ↑ and ↓. However, note that in neither case is **S** actually parallel to the z axis since S_z is smaller than S, as is clear from (9.4) and (9.5). This is the same thing that we saw with respect to **L**: Even when L_z has its maximum value, it is smaller than L, so that **L** is never exactly parallel to the z axis.

A complete specification of an electron's state of motion requires that one specify its spin orientation as well as its orbital motion. For example, for an electron in a hydrogen atom, the quantum numbers n, l, m specify the orbital motion, but for each choice of n, l, m, the spin can be either up or down, corresponding to $m_s = \frac{1}{2}$ or $-\frac{1}{2}$. It turns out that the energy of the H atom is almost completely independent of the spin orientation. Thus the allowed energies calculated in Chapter 8 are still correct, but there are twice as many independent states in each energy level as we had calculated formerly. The ground state, with $n = 1$ and $l = m = 0$, can have $m_s = \pm\frac{1}{2}$ and is therefore twofold degenerate. More generally, we saw that for the nth energy level there are n^2 possible values of l and m; for each of these there are two possible spin orientations. Therefore, the total degeneracy of the nth level is $2n^2$:

$$\text{(degeneracy of } n\text{th level in hydrogen)} = 2n^2 \tag{9.6}$$

When an electron's state of motion is completely specified, we say that the electron is in a definite **quantum state**, or just **state**; for example, we can speak of the quantum state identified by the four quantum numbers $n, l, m,$ and m_s in hydrogen. A specification of an electron's orbital motion, but not its spin orientation, is sometimes called an **orbital**; for example, we can speak of the orbital given by the three quantum numbers $n, l,$ and m. For each orbital, there are evidently two independent quantum states, corresponding to the two possible values of m_s.

Most of the elementary particles have a spin angular momentum. For example, both the proton and the neutron, like the electron, have spin half; that is, they have a spin angular momentum whose magnitude is given by (9.2), with $s = \frac{1}{2}$. In fact, all elementary particles have a spin angular momentum given by (9.2), although different particles may have different values of the spin quantum number s. Thus the photon is found to have $s = 1$ ("spin one"). For the pion, $s = 0$; that is, the pion has no spin angular momentum at all. For a particle called the Δ (delta), $s = \frac{3}{2}$ (see Chapter 18), and so on.

As far as we know, the electron is entirely elementary. That is, we have no evidence for any internal constituents of the electron. In particular, the spin angular momentum of the electron is not, as far as we know, the result of the internal motion of any "sub-elementary" particles. To emphasize this point, we sometimes say that its spin is an *intrinsic* property of the electron. By contrast, we now know that the proton is made up of more fundamental particles called quarks and gluons, and the proton's spin angular momentum is really just the vector sum of the angular momentum of these constituents. In a similar way, from the point of view of an atomic physicist, the atomic nucleus behaves just like a structureless "elementary" particle. The internal motions of its constituent protons and neutrons may give a nucleus some angular momentum, and this angular momentum is given by a formula just like (9.2). We often refer to this angular momentum as the nuclear "spin," even though we know that it is just the vector sum of the angular momenta of all the constituent protons and neutrons. In this chapter our focus is almost entirely on the electron, for which the spin is an intrinsic property with $s = \frac{1}{2}$.

9.3 Magnetic Moments

There is an enormous body of evidence for the electron's spin angular momentum. However, most of this evidence is indirect. In particular, much of the evidence for spin relates not to the angular momentum itself, but to the magnetic moment associated with any rotating electric charge. We must, therefore, review the concept of magnetic moment and describe its relation to the rotational motion of a charged particle. In this section and Section 9.4 we confine ourselves to the magnetic properties associated with the orbital motion of an electron. Then in Section 9.5 we describe the additional magnetic moment that results from the electron's spin. Finally, in Sections 9.6 and 9.7 we describe some phenomena in which the spin magnetic moment plays an important role.

We start by considering the magnetic properties of a classical point electron traveling around a nucleus in a circular orbit. An orbiting charge acts like a small current loop, and we know from classical electromagnetic theory that a current loop both produces a magnetic field and responds to an externally applied field. If a current i flowing around a small plane loop of area A is placed in a magnetic field $\mathbf{B}$, it experiences a torque $\boldsymbol{\Gamma}$ given by

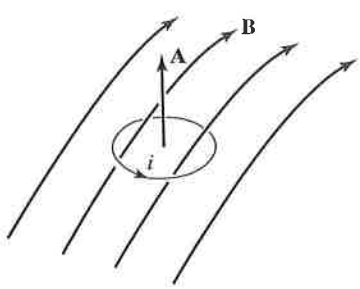

FIGURE 9.1
The current i flows around a loop specified by the vector $\mathbf{A}$, whose magnitude gives the loop's area and whose direction is perpendicular to the plane of the loop. A magnetic field $\mathbf{B}$ exerts a torque that tends to align $\mathbf{A}$ with $\mathbf{B}$.

$$\boldsymbol{\Gamma} = i\mathbf{A} \times \mathbf{B}$$

where the vector $\mathbf{A}$ has magnitude equal to the area A and is perpendicular to the plane of the loop as in Fig. 9.1. The sense of the vector $\mathbf{A}$ is given by the familiar right-hand rule: If you curl the fingers of your right hand around the loop in the direction of i, your thumb will point in the direction of $\mathbf{A}$.

It is usual to rewrite the torque $\boldsymbol{\Gamma} = i\mathbf{A} \times \mathbf{B}$ as

$$\boldsymbol{\Gamma} = \boldsymbol{\mu} \times \mathbf{B} \tag{9.7}$$

where the vector $\boldsymbol{\mu}$,

$$\boldsymbol{\mu} = i\mathbf{A} \tag{9.8}$$

is called the **magnetic moment** of the loop. The torque $\boldsymbol{\Gamma}$ tends to turn the loop so that $\boldsymbol{\mu}$ points in the same direction as the magnetic field $\mathbf{B}$.

Because of the torque (9.7), a current loop in a B field has a potential energy U that depends on the loop's orientation. To evaluate this energy, we recall that the work done by a torque Γ as it turns through an angle $d\theta$ is $\Gamma\, d\theta$.

The torque (9.7) has magnitude $\mu B \sin\theta$ and is in a direction to decrease θ. Thus the work done by B when the loop is brought to angle θ is

$$W = -\int \Gamma\, d\theta = -\mu B \int \sin\theta\, d\theta = \mu B \cos\theta + \text{constant} \qquad (9.9)$$

The potential energy U is defined as the negative of this work. Since the definition of potential energy always contains an arbitrary constant, it is customary to set the constant in (9.9) equal to zero, with the result that

$$U = -\mu B \cos\theta = -\boldsymbol{\mu}\cdot\mathbf{B} \qquad (9.10)$$

Notice that this potential energy is minimum when $\theta = 0$, with $\boldsymbol{\mu}$ pointing along $\mathbf{B}$ in stable equilibrium. (See Fig. 9.2.)

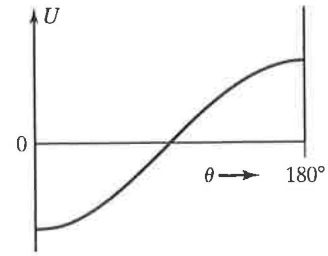

FIGURE 9.2
The potential energy $U = -\mu B \cos\theta$ of a magnetic moment $\boldsymbol{\mu}$ in a field $\mathbf{B}$ is minimum when $\theta = 0$ and $\boldsymbol{\mu}$ is parallel to $\mathbf{B}$.

Let us now consider the current loop produced by a (still classical) electron in circular orbit with radius r and period T. The current i is equal to the total charge passing any fixed point in unit time. The electron has charge of magnitude e, and speed $v = 2\pi r/T$. Therefore, the current has magnitude

$$i = \frac{e}{T} = e\frac{v}{2\pi r}$$

and the magnetic moment has magnitude

$$\mu = iA = \frac{ev}{2\pi r}\pi r^2 = \frac{1}{2}evr \qquad (9.11)$$

It is convenient to relate the magnetic moment μ to the angular momentum L. (Since μ and L both result from the electron's orbital motion, one might expect some simple relation between them.) Since the angular momentum has magnitude

$$L = m_e v r$$

(where m_e denotes the electron mass), we see from (9.11) that

$$\frac{\mu}{L} = \frac{e}{2m_e} \qquad (9.12)$$

We conclude that the ratio of μ to L — the so-called **gyromagnetic ratio** — is a constant that depends only on the charge and mass of the electron. Because the electron's charge is negative, the current is in a direction opposite to the electron's velocity, so that the vectors $\boldsymbol{\mu}$ and $\mathbf{L}$ are antiparallel. Thus, we can rewrite (9.12) in vector form as

$$\boldsymbol{\mu} = -\frac{e}{2m_e}\mathbf{L} \qquad (9.13)$$

We have derived the result (9.13) for the magnetic moment of a classical point electron in a circular orbit. In quantum mechanics it turns out that exactly the same expression correctly predicts the magnetic moment due to the orbital motion of an electron, provided that we use the correct quantum values for the magnitude and components of **L**. In particular, for a given magnitude $\sqrt{l(l+1)}\hbar$ we know that **L** has just $2l+1$ possible orientations. According to (9.13), the same is true of the magnetic moment $\boldsymbol{\mu}$ of an orbiting electron: For a given value of l, $\boldsymbol{\mu}$ has just $2l+1$ possible orientations.

Equation (9.13) gives the magnetic moment due to the orbital motion of an electron. As one might expect, there is an additional magnetic moment due to the electron's spinning motion. Before we discuss this spin magnetic moment, we use (9.10) and (9.13) to see how the energy of an atom changes when it is put in a magnetic field.

9.4 The Zeeman Effect

Pieter Zeeman
(1865–1943, Dutch)

At the suggestion of his teacher Lorentz, Zeeman investigated the effect of magnetic fields on atomic spectra. The results confirmed Lorentz's suspicion that atomic spectra are somehow connected to the motion of electrons in the atoms. Zeeman and Lorentz shared the 1902 Nobel Prize in physics for this work.

Because of the motion of their electrons, most atoms have a magnetic moment $\boldsymbol{\mu}$. Therefore, by applying a magnetic field **B**, one can change an atom's energy levels by an amount $-\boldsymbol{\mu}\cdot\mathbf{B}$ as in (9.10). This means that the energies of photons emitted and absorbed by the atom will change. That is, by putting an atom in a magnetic field, one can change its spectrum. This effect was first observed in 1896 by the Dutch physicist Pieter Zeeman and is called the **Zeeman effect**.

To simplify our discussion, we consider at first an atom in which the magnetic moments due to the electrons' spins cancel out. The simplest atom in which this can happen is helium, with its two electrons, and this is the atom that we consider. In many states of helium, including the ground state, the spins of the two electrons point in opposite directions so that the total magnetic moment due to the spins is zero. These states with zero total spin are called **singlet states.** Furthermore, it turns out that in all the states of helium, one of the electrons has zero orbital angular momentum. Therefore, the total magnetic moment for any of the singlet states of helium is just the moment $\boldsymbol{\mu} = -(e/2m_e)\mathbf{L}$ due to the orbital motion of the second electron.

In the absence of a magnetic field, the helium atom has an energy that we call E_0, and its angular momentum (the orbital momentum **L** of the second electron) has a magnitude given by the quantum number $l = 0, 1, 2, \cdots$. The angular momentum **L** has $2l+1$ possible different orientations, corresponding to the $2l+1$ possible values of $L_z = m\hbar$, with $m = l, l-1, \cdots, -l$. In the absence of a magnetic field, the energy is the same for all of these states, and the level E_0 is $(2l+1)$-fold degenerate.

Suppose now that we apply a magnetic field **B** to our atom. According to (9.10) this will change the atom's energy by the amount $-\boldsymbol{\mu}\cdot\mathbf{B}$, which depends on the orientation of $\boldsymbol{\mu}$. Now $\boldsymbol{\mu}$ is given by (9.13) and has $2l+1$ different possible orientations. Therefore, we can anticipate *not only that the energy will change as a result of the magnetic field*, but that *it will change by a different amount for each of the* $2l+1$ *different orientations*. That is, by applying a magnetic field, we remove the $(2l+1)$-fold degeneracy of the original energy level.

The size of the energy shift due to the magnetic field is easily calculated: With the field switched on, we denote the total energy by $E = E_0 + \Delta E$, where the shift ΔE is given by (9.10) and (9.13) as

$$\Delta E = -\boldsymbol{\mu}\cdot\mathbf{B} \tag{9.14}$$

$$= \left(\frac{e}{2m_e}\right)\mathbf{L}\cdot\mathbf{B} \tag{9.15}$$

If we choose our z axis in the direction of the applied field $\mathbf{B}$, then (9.15) simplifies to

$$\Delta E = \left(\frac{e}{2m_e}\right) L_z B$$

or, since the possible values of L_z are $m\hbar$,

$$\Delta E = \left(\frac{e\hbar}{2m_e}\right) mB \qquad (9.16)$$

As anticipated, the magnetic field changes the atom's energy by an amount that depends on the quantum number m. This is why m is often called the **magnetic quantum number**, and it explains the traditional choice of the letter m.

Comparing (9.16) with (9.14) (and remembering that the quantum number m is dimensionless), we see that the quantity in parentheses, $(e\hbar/2m_e)$, must have the dimensions of a magnetic moment. (You should check this directly; see Problem 9.14.) In atomic physics this quantity is a convenient unit for magnetic moments and is called the **Bohr magneton** μ_B, with the value

$$\mu_B = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24}\ \text{A} \cdot \text{m}^2 \qquad (9.17)$$

In terms of μ_B, we can rewrite (9.16) in the compact form

$$\Delta E = m\mu_B B \qquad (9.18)$$

Since m can have the $2l + 1$ values $l, l - 1, \cdots, -l$, we see that the $2l + 1$ states of the original degenerate level now have energies that are equally spaced, an energy $\mu_B B$ apart:

$$(\text{separation of adjacent levels}) = \mu_B B \qquad (9.19)$$

The result (9.19) shows that the dimensions of μ_B can also be expressed as energy/(magnetic field); that is, the unit $\text{A} \cdot \text{m}^2$ in (9.17) can be replaced by joules/tesla. If we convert the joules to eV, we get the useful result

$$\mu_B = 5.79 \times 10^{-5}\ \text{eV/T} \qquad (9.20)$$

According to (9.19), this means that a field of 1 tesla leads to a separation of adjacent levels by 5.79×10^{-5} eV — a very small separation on the scale of normal atomic levels, which are typically a few eV apart.

Example 9.1

A helium atom is in one of its singlet states (with the two spins antiparallel and hence no spin magnetic moment). One of its electrons is in an s state ($l = 0$) and the other a d state ($l = 2$). The atom is placed in a magnetic field, $B = 2$ T (by normal laboratory standards a fairly strong field). By how much does the magnetic field change the atom's energy?

The shift in energy is given by (9.18) as $\Delta E = m\mu_B B$, where, since $l = 2$, the quantum number m can have any of the five values $m = 2, 1, 0,$

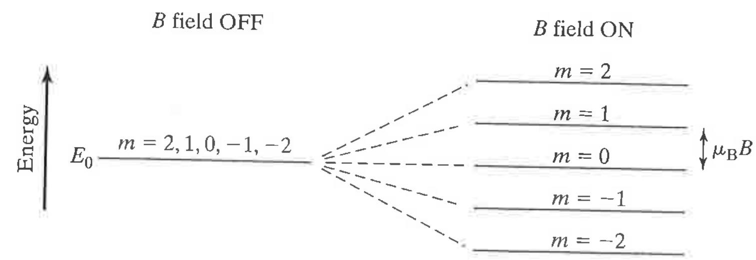

FIGURE 9.3
The Zeeman effect. In the absence of a magnetic field, an atomic level with $l = 2$ (and no spin magnetic moment) is fivefold degenerate with energy E_0. When B is switched on, the level splits into a multiplet of five equally spaced levels with separation $\mu_B B$.

−1, −2. If the atom is in the state with $m = 0$, its energy is unaltered. If it is in any of the other four states, its energy is shifted as shown in Fig. 9.3. The five resulting energy levels are said to form a multiplet and are evenly spaced above and below the original E_0, with separation $\mu_B B$. With $B = 2$ tesla, the separation of adjacent levels is

$$\mu_B B = \left(5.79 \times 10^{-5} \frac{\text{eV}}{\text{T}}\right) \times (2\text{ T}) = 1.2 \times 10^{-4}\text{ eV}$$

We see from this example that even a relatively strong field of a few teslas produces a very small separation of energy levels. Nevertheless, this small splitting of each level into several levels results in an observable splitting of the spectral lines of the light emitted and absorbed by the atom. To illustrate this effect — the Zeeman effect — we consider again the helium atom. Specifically, we consider the ground state and one of the low lying excited states, with energy 21.0 eV above the ground state.

In both of these states of the helium atom, the two electron spins are antiparallel and the resultant spin magnetic moment is zero. Thus the shift in energy produced by a magnetic field is correctly given by (9.18) as

$$\Delta E = m\mu_B B$$

In the ground state both electrons have $l = 0$. Thus, the only possible value of m is $m = 0$, and the shift ΔE is zero. That is, the energy of the ground state is unchanged by a magnetic field. In the excited state, one electron has zero orbital angular momentum while the other has $l = 1$. With $l = 1$, the possible values of m are $m = 1, 0$, and -1, and the magnetic field splits this level into three equally spaced levels, a distance $\mu_B B$ apart, as shown in Fig. 9.4.

Let us now consider transitions in which a helium atom drops from the excited state just described to the ground state and emits a photon. In the

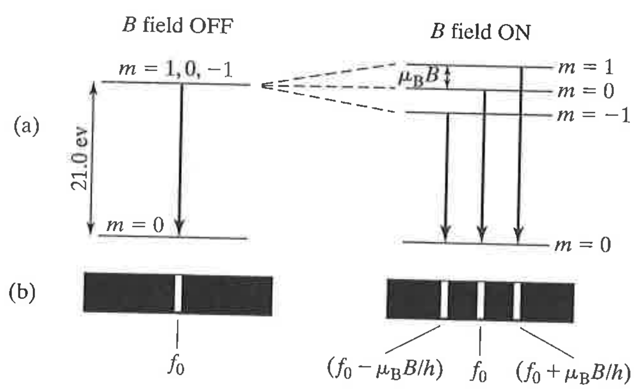

FIGURE 9.4
(a) The ground state ($l = 0$) and one of the excited levels ($l = 1$) of helium. When a magnetic field is applied, the upper level splits into three, while the ground state is unaffected. (The splitting of the levels is greatly exaggerated since, even in the strongest magnetic fields obtainable in a lab — about 40 T — the separation is only $\mu_B B \approx 2 \times 10^{-3}$ eV.) **(b)** With the magnetic field on, there are three distinct transitions possible and hence three distinct spectral lines, as shown on the right.

absence of a magnetic field, both levels have unique energies separated by 21.0 eV, and the photon has energy $E_\gamma = 21.0$ eV and frequency $f_0 = E_\gamma/h$. Thus a spectrometer would reveal a single spectral line with frequency f_0, as indicated at the bottom left of Fig. 9.4. If we now apply a magnetic field B, the upper level splits into three closely spaced levels. Therefore, there are now three possible transitions with three slightly different energies, as indicated by the three downward arrows on the right side of Fig. 9.4. In a gas of excited helium there would normally be atoms in all three levels. Therefore, all three transitions would occur, and a spectrometer would now reveal a triplet of three closely spaced spectral lines with frequencies $f_0 + \mu_B B/h$, f_0, and $f_0 - \mu_B B/h$, as shown at the bottom right of Fig. 9.4.

The Zeeman effect was discovered in 1896, well before the development of quantum mechanics. Naturally, attempts were made to explain it on the basis of classical mechanics. As it happens, the classical theory of the Zeeman effect gives the correct answers for any two atomic states whose spin magnetic moments are zero. Thus a Zeeman splitting like that shown in Fig. 9.4(b) agreed with classical predictions and came to be called the **normal Zeeman effect.*** Unfortunately (for classical mechanics), the effect of a magnetic field on many atoms was found to be much more complicated than this and did not agree with the classical theory. As we will see, these more complicated shifts, which were called **anomalous Zeeman splittings**, involve the spin magnetic moment of the electron and were among the first indications that the electron has a spin.

Example 9.2

What is the wavelength λ_0 of the transition shown on the left of Fig. 9.4? If a magnetic field of 2 T is applied to the helium atom, what are the shifts $\Delta\lambda$ of the outer two spectral lines on the right of Fig. 9.4(b)?

With $B = 0$ the two states are 21.0 eV apart in energy and the emitted photon has $E_\gamma = 21.0$ eV. The corresponding wavelength is

$$\lambda_0 = \frac{hc}{E_\gamma} = \frac{1240\ \text{eV}\cdot\text{nm}}{21.0\ \text{eV}} = 59.0\ \text{nm}$$

which is in the far ultraviolet.

If we switch on a magnetic field, the upper level splits into the three equally spaced levels shown in Fig. 9.4, separated by energy $\mu_B B$:

$$\begin{aligned}(\text{separation of adjacent levels}) &= \mu_B B \\ &= \left(5.79 \times 10^{-5} \frac{\text{eV}}{\text{T}}\right) \times (2\ \text{T}) \\ &= 1.2 \times 10^{-4}\ \text{eV}\end{aligned}$$

The $m = 0$ state has the same energy as when $B = 0$, and the wavelength of photons emitted from this state is unchanged. The energy of photons emitted

*The example of Fig. 9.4(b) involved a transition from $l = 1$ to $l = 0$ and gave a splitting into three spectral lines. One might imagine that higher l values would lead to more than three lines; in fact, however, the normal Zeeman effect always produces exactly three lines (Problem 9.17).

from the states with $m = \pm 1$ is changed by $\Delta E_\gamma = \pm \mu_B B$. Since this is small, the shift in their wavelength is well approximated as

$$\begin{aligned}\Delta\lambda \approx \frac{d\lambda}{dE_\gamma}\Delta E_\gamma &= -\frac{hc}{E_\gamma^2}\Delta E_\gamma \\ &= -\frac{1240\ \text{eV}\cdot\text{nm}}{(21.0\ \text{eV})^2} \times (\pm 1.2 \times 10^{-4}\ \text{eV}) \\ &= \mp 3.4 \times 10^{-4}\ \text{nm}\end{aligned} \tag{9.21}$$

The Zeeman shift of wavelength is so small that the earliest observations could not distinguish the separate spectral lines. At first, all that was detected was a broadening of the original single line; but later experiments with better resolution showed that the line was indeed split into several separate lines. Today, spectrometers can resolve splittings of order 10^{-8} nm, and the Zeeman shifts can be measured very accurately. An important modern application is to measure the splitting of an identified spectral line and hence to find an unknown magnetic field. This is especially useful in astronomy since the magnetic fields of the sun and stars cannot be measured directly.

9.5 Spin Magnetic Moments

We have seen that as an atomic electron orbits around the nucleus, it produces a magnetic moment given by (9.13) as

$$\boldsymbol{\mu}_{\text{orb}} = -\frac{e}{2m_e}\mathbf{L} \tag{9.22}$$

If we visualize the electron as a tiny, rigid ball of charge spinning on its axis, we would expect this spinning motion to produce an additional spin magnetic moment. Each piece of the electron would be carried in a circular path around the axis and hence constitute a small current loop. Each such loop would produce a magnetic moment, and the sum of all these moments would be the total spin magnetic moment $\boldsymbol{\mu}_{\text{spin}}$. Since $\boldsymbol{\mu}_{\text{spin}}$ would be proportional to the angular velocity $\boldsymbol{\omega}_{\text{spin}}$, which in turn is proportional to the spin angular momentum $\mathbf{S}$, we would expect to find $\boldsymbol{\mu}_{\text{spin}} \propto \mathbf{S}$ or

$$\boldsymbol{\mu}_{\text{spin}} = -\gamma\mathbf{S} \tag{9.23}$$

where γ is a constant called the **spin gyromagnetic ratio.** [We have put a minus sign in (9.23) because the electron's charge is negative, and $\boldsymbol{\mu}_{\text{spin}}$ and $\mathbf{S}$ are in opposite directions.] In the case of the orbital motion, the gyromagnetic ratio is seen from (9.22) to be $e/2m_e$. The spin gyromagnetic ratio would not necessarily have this same value, since it would depend on the distributions of charge and mass within the electron. If the charge were concentrated farther out than the mass, then μ_{spin}/S would be relatively large; if the charge were concentrated nearer the center, μ_{spin}/S would be smaller.

The classical picture of the electron as a rigid spinning ball of charge is not strictly correct. For example, if the radius of the ball is taken consistent with modern observations, the equatorial speed turns out to be greater than c, which is impossible (Problem 9.8). Nevertheless, the conclusions that there should be a magnetic moment with the general form (9.23) and that the spin gyromagnetic ratio does not necessarily have the same value as the orbital ratio $e/2m_e$ are both correct. Experiment shows that there is a magnetic moment with the form (9.23) and that the spin gyromagnetic ratio γ is e/m_e, just twice* the value of the orbital ratio, $e/2m_e$; that is,

$$\boldsymbol{\mu}_{\text{spin}} = -\frac{e}{m_e}\mathbf{S} \tag{9.24}$$

The total magnetic moment of any electron is just the sum of its orbital and spin moments

$$\boldsymbol{\mu}_{\text{tot}} = \boldsymbol{\mu}_{\text{orb}} + \boldsymbol{\mu}_{\text{spin}} = -\frac{e}{2m_e}(\mathbf{L} + 2\mathbf{S}) \tag{9.25}$$

As we describe in the next two sections, much of the evidence for the electron's spin comes from the repeated success of the formula (9.25) in explaining a wide variety of experimental results.

The suggestion that the electron has a spin angular momentum and a corresponding magnetic moment [given, as we now know, by (9.24)] is generally credited to the Dutch physicists Samuel Goudsmit and George Uhlenbeck (1925). Their suggestion was based on an analysis of the anomalous Zeeman effect (which we describe in Section 9.6) and of the fine structure in atomic spectra (Section 9.7). However, it is worth mentioning that similar suggestions had been made by other physicists. In particular, Arthur Compton had suggested that a spin magnetic moment for the electron could possibly explain the phenomenon of ferromagnetism, a suggestion that later proved to be correct.

George Uhlenbeck and Samuel Goudsmit

(1900–1988, Dutch, American)
(1902–1978, Dutch, American)

Uhlenbeck (center) and Goudsmit (right) are shown here with colleague Oskar Klein (left). In 1925 Goudsmit and Uhlenbeck, while both graduate students at Leiden, showed that several puzzles in atomic spectra could be explained if the electron was assumed to have a spin angular momentum with quantum number $s = \frac{1}{2}$. Both moved to the United States in 1927, and both worked at Michigan and then MIT. Goudsmit became editor of the *Physical Review*.

9.6 The Anomalous Zeeman Effect

When an atom is placed in a magnetic field, we have seen that its energy levels undergo small shifts and individual levels get split into several closely spaced levels. This results in a splitting of the spectral lines into closely spaced "multiplets" of lines — an effect known as the Zeeman effect.

In Section 9.4 we calculated in detail the Zeeman splitting for the so-called singlet states of the helium atom (the states in which the two spin magnetic moments cancel out and can therefore be ignored). The results of those calculations are correct for any atomic state in which the spin magnetic moments cancel. In general, however, the Zeeman effect does *not* agree with the splittings calculated in Section 9.4, but does agree with a corresponding calculation using the correct magnetic moment (9.25), including both orbital and spin moments. For historical reasons, the splitting of levels and spectral lines is called the *normal Zeeman effect* in those cases where spin has no effect and called the *anomalous Zeeman effect* in those cases where spin does contribute.

The correct calculation of the anomalous Zeeman splitting is quite complicated, depending as it does on both orbital and spin moments. To simplify our discussion, we consider here the simple case of a hydrogen atom in a state with no orbital angular momentum, that is, an s state, with $l = 0$. If the electron had no spin, then, with $l = 0$, the atom would have no magnetic moment at all,

*Precise measurements show that the spin ratio is actually not exactly twice the orbital value (instead of 2, the factor is 2.0023). The observed value is successfully predicted by the relativistic quantum theory called quantum electrodynamics.

and would be completely unaffected by a magnetic field. In fact, of course, the electron does have spin and, even though $l = 0$, there is a magnetic moment

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{\text{spin}} = -\frac{e}{m_e}\mathbf{S} \tag{9.26}$$

When a magnetic field **B** (in the z direction) is switched on, the energy changes by an amount

$$\Delta E = -\boldsymbol{\mu} \cdot \mathbf{B} = \frac{e}{m_e} S_z B$$

Since the possible values of S_z are

$$S_z = \pm\tfrac{1}{2}\hbar$$

it follows that

$$\Delta E = \pm\frac{e\hbar}{2m_e}B = \pm\mu_B B \tag{9.27}$$

If the electron has spin up, its energy is raised by $\mu_B B$; if it has spin down, its energy is lowered by the same amount. The resulting separation of the two levels is therefore

$$(\text{separation of levels}) = 2\mu_B B \tag{9.28}$$

Notice that this separation is twice the value (9.19) predicted for the normal Zeeman effect; this is because the spin gyromagnetic ratio is twice the orbital ratio.

We conclude that any $l = 0$ level in hydrogen should be split into two neighboring levels by a magnetic field. This splitting is sketched in Fig. 9.5. That these levels are observed to split into two levels is strong evidence that the electron has a spin $s = \frac{1}{2}$ (and hence just two possible orientations). The agreement of the observed level separation with (9.28) confirms the expression (9.26) for $\boldsymbol{\mu}_{\text{spin}}$.

The Zeeman effect has been observed in many different levels of dozens of different atoms, and in all cases the results confirm that the electron's total magnetic moment is given by (9.25) as $-(e/2m_e)(\mathbf{L} + 2\mathbf{S})$, that the angular momentum vector **S** has magnitude $\sqrt{s(s+1)}\hbar$ with s always equal to $\frac{1}{2}$, and that S_z has the two possible values $\pm\frac{1}{2}\hbar$.

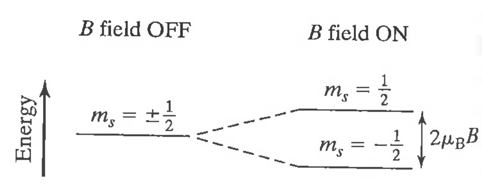

FIGURE 9.5
Any s level in hydrogen is split by a B field into two levels because of the electron's spin magnetic moment. The separation of the levels is $2\mu_B B$.

9.7 Fine Structure*

*Though important, this material will not be needed later and can be omitted without loss of continuity.

The Zeeman effect is a splitting of atomic energy levels caused by an *externally applied* magnetic field. In most atoms there is a permanent *internal* magnetic field due to the motion of the charges inside the atom. Even when

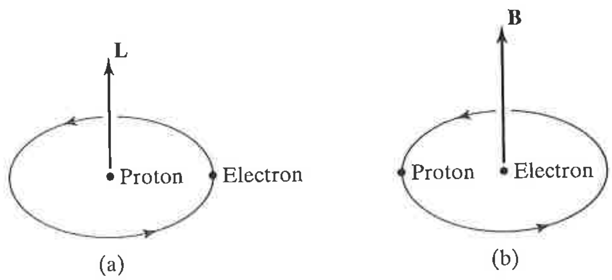

FIGURE 9.6
(a) In the proton's rest frame the electron orbits around the proton, with orbital angular momentum **L**. (Because the proton is so much heavier than the electron, this frame is very close to the rest frame of the atom as a whole, and is the frame usually considered.) **(b)** In the electron's rest frame, the proton orbits around the electron. The sense of the orbit is the same in both pictures. Since the proton's charge is positive, it produces a magnetic field **B** (given by the right-hand rule) in the direction shown. Therefore, **B** is in the same direction as **L**.

there is no external magnetic field, this internal field can cause a small splitting of the energy levels and, hence, of the atomic spectrum. These splittings due to the internal magnetic field are called **fine structure**. As an illustration, we describe briefly the fine structure of hydrogen, many of whose spectral lines were found to be *doublets* consisting of two closely spaced lines.

Let us consider the states of a hydrogen atom with some definite energy (given by quantum number n) and definite nonzero orbital angular momentum (given by quantum number l). We can understand the fine structure of these states from the following semiclassical argument: In the rest frame of the electron, the proton orbits around the electron, as shown in Fig. 9.6. Therefore, the electron finds itself in the magnetic field produced by the current loop of an orbiting positive charge. This field is proportional to the orbital frequency of the proton (as seen in the electron's rest frame), which in turn is proportional to the orbital angular momentum **L** of the electron, as seen in the proton's rest frame. Therefore, the electron sees a field **B** that is proportional to **L**:

$$\mathbf{B} \propto \mathbf{L} \tag{9.29}$$

As can be seen in Fig. 9.6, the direction of **B** is the same as that of **L**. Therefore, the constant of proportionality in (9.29) is positive.

Since the electron has a magnetic moment* $\boldsymbol{\mu}_{\text{spin}} = -(e/m_e)\mathbf{S}$, the magnetic field **B** gives it an additional energy

$$\Delta E = -\boldsymbol{\mu}_{\text{spin}} \cdot \mathbf{B} = \frac{e}{m_e}\mathbf{S} \cdot \mathbf{B} \propto \mathbf{S} \cdot \mathbf{L} \tag{9.30}$$

Because it is proportional to $\mathbf{S} \cdot \mathbf{L}$, this energy is often described as the **spin-orbit** energy. We will see that the spin-orbit energy is usually very small by atomic standards. (Typically, it is a very small fraction of an eV.) This means that the hydrogen energy levels calculated in Chapter 8, which ignored all effects of spin, are still an excellent approximation. Nevertheless, the correction (9.30) does cause a small splitting of the levels, as we now show.

According to (9.30), the electron has a magnetic energy that depends on the orientation of **S** relative to **L**. The spin **S** can have two possible orientations relative to any definite direction. Therefore, the term $\mathbf{S} \cdot \mathbf{L}$ in (9.30) can have two possible different values. This means that the states with any particular values of n and l actually belong to two slightly different energy levels. Those states in which **S** is parallel to **L** have a slightly higher energy; those in which **S** is antiparallel to **L** are slightly lower.

The argument just given applies to any state with $l \neq 0$. If $l = 0$, then since $\mathbf{B} \propto \mathbf{L}$, it follows that the B field seen by the electron is zero and hence that there is no spin-orbit splitting for s states.

*We need consider only the spin magnetic moment because the orbital momentum is always zero in the electron's rest frame.

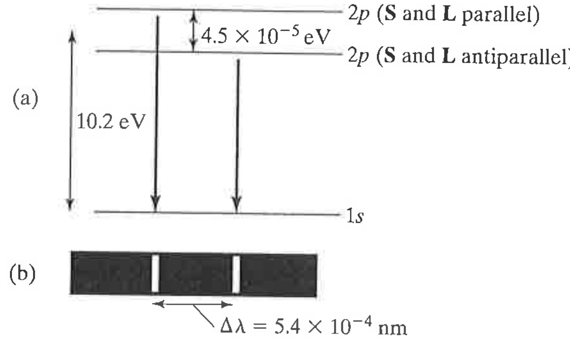

FIGURE 9.7

Fine structure in hydrogen. **(a)** The $1s$ states have a unique energy, while the $2p$ states belong to two slightly different energy levels, those with **S** parallel to **L** being slightly higher. (The separation of these levels is exaggerated by a factor of 50,000.) **(b)** Therefore, transitions from $2p$ to $1s$ involve photons with two slightly different energies and produce a doublet of spectral lines, as shown.

The splitting of each level (with $l \neq 0$) into two levels implies a corresponding splitting of the spectral lines of hydrogen. As an example, let us consider transitions in which a hydrogen atom in one of its $2p$ states drops to the ground state. We have seen that the ground state is not split by the spin-orbit interaction, whereas the $2p$ states are split into two levels. Thus the energy-level diagram for these states is as shown in Fig. 9.7, and there are two different possible photon energies. Therefore, the transitions from $2p$ states to the ground state produce a doublet of spectral lines, as shown in Fig. 9.7(b).

To estimate the separation of these two lines, we must first find the magnetic field B of the orbiting proton, as seen by the $2p$ electron. A straightforward calculation (Problem 9.21) shows that

$$B \approx 0.39\ \text{T} \tag{9.31}$$

According to (9.28), this implies that the $2p$ levels are separated by an energy,

$$\begin{aligned}(\text{separation of } 2p \text{ levels}) &= 2\mu_{\text{B}}B \\ &\approx 2 \times \left(5.8 \times 10^{-5}\frac{\text{eV}}{\text{T}}\right) \times (0.39\ \text{T}) \\ &\approx 4.5 \times 10^{-5}\ \text{eV}\end{aligned} \tag{9.32}$$

This is extremely small compared to the distance between the $2p$ and $1s$ levels, which is $(13.6 - 3.4) = 10.2$ eV. Therefore, the difference in wavelengths of the emitted photons is approximately

$$\begin{aligned}\Delta\lambda \approx \left|\frac{d\lambda}{dE_\gamma}\right|\Delta E_\gamma &= \frac{hc}{E_\gamma^{\,2}}\Delta E_\gamma \\ &= \frac{1240\ \text{eV}\cdot\text{nm}}{(10.2\ \text{eV})^2} \times (4.5 \times 10^{-5}\ \text{eV}) \\ &= 5.4 \times 10^{-4}\ \text{nm}\end{aligned}$$

as indicated in Fig. 9.7.

It should be emphasized that in our discussion of fine structure we have treated the electron as a classical orbiting particle. Also, we have consistently treated the hydrogen atom nonrelativistically, and although this is certainly an excellent approximation, there are small corrections required to allow for relativity. It turns out that these relativistic corrections are of the same order of magnitude as the spin-orbit energy discussed here. (See Problems 9.21 to 9.23.) Thus a correct analysis of the fine structure of hydrogen needs to be fully quantum mechanical and to take account of relativity. Under these conditions our calculation of the splittings can only be regarded as an order-of-magnitude estimate. That the answer (9.32) is correct to two significant figures is simply a happy accident. Nevertheless, all of our general conclusions are qualitatively correct.

9.8 Magnetic Resonance Imaging (MRI)*

*Though important, this material will not be needed later and can be omitted without loss of continuity.

In the 1970s, scientists from several universities and industries developed a new medical diagnostic technique called **magnetic resonance imaging** (MRI) or nuclear magnetic resonance (NMR). Today, it is a common and safe imaging technique, producing pictures of internal organs without exposing the patient to ionizing radiation, such as conventional X-rays (see Fig. 9.8). Although more expensive and slower than X-rays, MRI is better able to distinguish between different types of soft tissues. This remarkable technique involves measuring the concentration of *proton* spins in the patient's body.

We begin with a short discussion of proton spin. As we mentioned at the end of Section 9.2, the proton, like the electron, is a spin-half particle. Like the electron, the proton has S_z equal to $\pm\frac{1}{2}\hbar$. Also like the electron, it has a magnetic moment related to its spin by an equation similar to (9.23),

$$\boldsymbol{\mu}_{\text{proton}} = \gamma_{\text{proton}}\mathbf{S} \tag{9.33}$$

There are two differences between this equation and (9.23) for electrons: (1) In this equation there is no minus sign because the proton is positive. (2) The gyromagnetic factor for protons is some 660 times smaller than for electrons, resulting in a proton magnetic moment 660 times smaller than the electron moment. The smallness of the nuclear gyromagnetic factor is due to the largeness of the proton mass: Just as the electron gyromagnetic ratio is given by $\gamma = e/m_e$, the proton gyromagnetic ratio is of order* $\gamma \approx e/m_p$, where m_p is the proton mass.

Just as with the electron, a proton placed in a magnetic field can exist in one of two different energy states, corresponding to the magnetic moment and

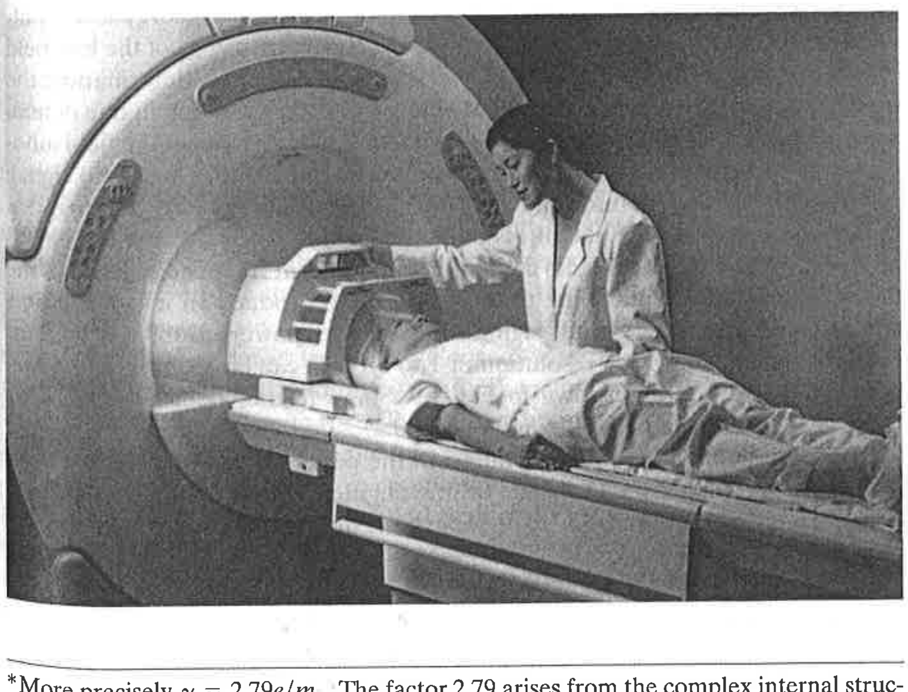

FIGURE 9.8

A magnetic resonance imaging (MRI) apparatus. The large structure contains a superconducting magnet producing a 1-tesla magnetic field. The smaller cylindrical structure surrounding the head of the patient is the pick-up coil, which detects the radio-frequency signal from the protons in the patient's body.

*More precisely, $\gamma = 2.79e/m_p$. The factor 2.79 arises from the complex internal structure of the proton, which consists of 3 quarks.

B field being aligned or anti-aligned. Looking at (9.33) (and dropping the "proton" subscripts), we see that the energies of the two levels are given by

$$-\mu_z B = -\gamma S_z B = \pm\tfrac{1}{2}\gamma\hbar B \tag{9.34}$$

and the energy-level separation is $\Delta E = \gamma\hbar B$. Transitions between these two levels can be induced by the absorption or emission of electromagnetic radiation of angular frequency ω such that $\hbar\omega = \Delta E = \hbar\gamma B$. When the magnetic field is a few tesla, this frequency happens to be in the range of radio frequencies, which is roughly 1 to 100 MHz. Specifically, for a 1-tesla field, the frequency of this radiation is $f = \omega/2\pi = 42.6$ MHz (see Problem 9.25).

Humans, of course, are full of protons. Every water molecule (H_2O) in our body contains two hydrogen nuclei, which are protons. (We will see that the other nuclei, such as oxygen, are not normally involved in MRI.) The human patient, with all her proton spins, is placed in a very strong magnetic field, typically $B = 1$ or 2 tesla. Because of this B field, every proton in the patient's body exists in one of two energy levels, with spin up or spin down. In an MRI apparatus, a transmitter coil surrounding the patient produces a pulse of radio waves at the correct frequency, the "resonant frequency," to induce transitions between the spin levels, generating a nonequilibrium population of spin up versus spin down in the patient. As the spins spontaneously relax back to equilibrium, the transitions between the two levels produce electromagnetic fields with a specific radiofrequency precisely proportional to the applied field, $\omega = \gamma B$. This electromagnetic signal is detected by a receiver coil around the patient,* and the frequency of this signal gives a very accurate measurement of the magnetic field in which the patient has been placed.

So far we have described a very expensive way to use the protons in a human to measure a magnetic field. Now comes the clever part. The magnetic field in which the patient lies is not uniform; it is intentionally made inhomogeneous so that the field varies linearly from one side of the patient to the other. The protons in the high-field side of the patient have greater energy-level splitting and produce higher-frequency radio emissions than protons of the low-field side. Thus, a particular proton's radio signal contains position information; the signal's frequency is exactly proportional to the proton's position in one dimension, the axis along which the field varies. By switching the axis of the field inhomogeniety among the x, y, and z directions while monitoring the radio signal, one can collect enough information to construct a three-dimensional image of the proton density in the patient. The details are quite messy: The receiver coil around the patient is picking up a multitude of different frequencies from all the protons at once. Disentangling the frequencies to produce an image requires very sophisticated software and significant computer power.† But the results are remarkable; images with a resolution of 1 to 0.1 mm can be acquired in several minutes. An example is shown in Fig. 9.9. Some further details of the MRI process are presented in Problems 9.26 and 9.27.

Of the many other types of nuclei in the patient's body, such as O, N, C, and so forth, each has its own value of spin angular momentum, and each has a characteristic gyromagnetic ratio and magnetic moment. In some cases, such as

* Often, a single coil acts as both transmitter and receiver. This transceiver coil is separate from the superconducting coils of the magnet producing the large B field.

† When MRI units were first used in the late 1970s, large mainframe computers were needed to handle the image processing calculations. Today (2003), one good PC can handle the job.

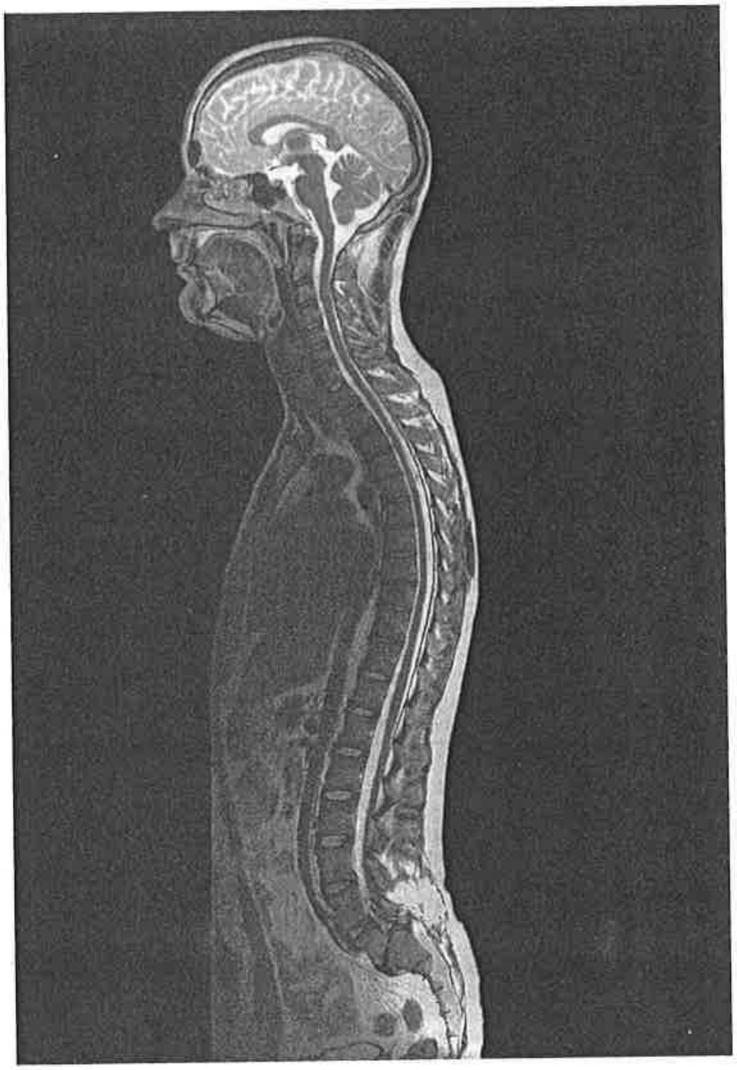

FIGURE 9.9
MRI scan of a head and spine.

^{16}O, the spin happens to be zero, but in many cases the magnetic moment is nonzero and these nuclei can produce an MRI signal. However, all these other nuclei possess gyromagnetic ratios γ and corresponding frequencies $\omega = \gamma B$ that are different from that of the proton. The transmitter/receiver electronics in an MRI apparatus is tuned to induce transitions only for nuclei with frequencies near the proton frequency. Consequently, the presence of the other nuclei does not affect the proton signal. Of the many types of nuclei in humans, hydrogen nuclei are chosen for MRI imaging because they are both numerous and have a large magnetic moment, so they provide the strongest signal.

The single most expensive component of an MRI apparatus is the magnet. Producing such a well-controlled field of 1 or 2 tesla over a volume large enough to insert a human is a major engineering accomplishment. The magnet used in an MRI apparatus has superconducting windings cooled to a few degrees above absolute zero with the use of liquid helium. (See Chapter 14 for a discussion of superconductivity.) Although such superconducting magnets are quite expensive (roughly a million dollars), they are cheaper to operate than conventional iron-core electromagnets, which would require enormous amounts of electrical power and cooling water to produce a magnetic field of comparable strength. The commercial production of such reliable, affordable superconducting magnets was achieved in the late 1970s and was a major milestone on the road to making MRI a practical tool.

CHECKLIST FOR CHAPTER 9

CONCEPT	DETAILS
Spin angular momentum of the electron	quantum number $s = \frac{1}{2}$ magnitude $S = \sqrt{s(s+1)}\hbar = \sqrt{3}\hbar/2$ (9.2) $S_z = \pm\frac{1}{2}\hbar$ (9.5)
Classical magnetic moment $\boldsymbol{\mu}$ potential energy	$\boldsymbol{\mu} = i\mathbf{A}$ (9.8) $U = -\boldsymbol{\mu}\cdot\mathbf{B}$ (9.10)
Gyromagnetic ratio γ	ratio of magnetic moment μ to angular momentum (Secs. 9.3 and 9.5)
Electron's orbital magnetic moment	$\boldsymbol{\mu}_{orb} = -(e/2m_e)\mathbf{L}$ (9.13)
The Bohr magneton	convenient unit for atomic magnetic moments, $\mu_B = e\hbar/(2m_e)$ (9.17)
Zeeman splitting	$\Delta E = m\mu_B B$ (9.18), energy shift of orbital magnetic moment in a field
Electron's spin magnetic moment	$\boldsymbol{\mu}_{spin} = -(e/m_e)\mathbf{S}$ (9.24)
Fine structure★	$\Delta E = -\boldsymbol{\mu}_{spin}\cdot\mathbf{B} \propto \mathbf{S}\cdot\mathbf{L}$, where $\mathbf{B}$ is due to orbital motion of proton as seen by electron (9.30)
Magnetic resonance imaging (MRI or NMR)★	proton spin in a magnetic field, radiofrequency emissions of frequency $\omega = \gamma B$ (Sec. 9.8)

PROBLEMS FOR CHAPTER 9

SECTION 9.2 (Spin Angular Momentum)

9.1 • One can visualize the quantized values of the spin angular momentum $\mathbf{S}$ with a semiclassical "vector model," as described in Section 8.6 for the orbital angular momentum $\mathbf{L}$. In particular, the quantization of S_z requires that the vector $\mathbf{S}$ must lie on certain cones, like the one sketched in Fig. 8.15. **(a)** Make a sketch similar to Fig. 8.14 showing the two possible orientations of $\mathbf{S}$ for an electron. **(b)** What is the angle between $\mathbf{S}$ and the z axis for these two states?

9.2 • There exist subatomic particles with spin magnitudes different from that of the electron. However, in all cases they obey the same rules: The magnitude of $\mathbf{S}$ is $\sqrt{s(s+1)}\hbar$, where s is a fixed number, integer or half-integer; and the possible values of S_z are $m_s\hbar$, where m_s has the values $s, s-1, \cdots, -s$. **(a)** For a particle with $s = \frac{3}{2}$, how many different values of S_z are there, and what are they? **(b)** Draw a vector model diagram similar to Fig. 8.14 showing the possible orientations of $\mathbf{S}$. **(c)** What is the minimum possible angle between $\mathbf{S}$ and the z axis?

9.3 • Answer the same questions as in Problem 9.2, but for a particle with spin quantum number $s = 1$.

9.4 • Make a table showing the values of the four quantum numbers n, l, m, m_s for each of the 18 states of the hydrogen atom with energy $E = -E_R/9$.

9.5 • Make a table showing the values of the four quantum numbers n, l, m, m_s and the energies for each of the 10 lowest-lying quantum states (not energy levels) of a hydrogen atom.

9.6 • Compute the ratio S/L of the magnitudes of the spin angular momentum to the orbital angular momentum for **(a)** an s electron, **(b)** a p electron, **(c)** a d electron, **(d)** an f electron. Note that for all but the s electron, the size of the spin angular momentum is of the same order of magnitude as the orbital angular momentum.

9.7 •• **(a)** Write an expression for the magnitude of orbital angular momentum L_{orb} of the earth due to its orbital motion about the sun. **(b)** Assume that the earth has uniform mass density, and derive an expression for the spin angular momentum L_{spin} of the earth due to its rotation about its axis. **(c)** Show that for the earth, the ratio L_{spin}/L_{orb} is given by the expression $2R_E{}^2T_{year}/(5R_{SE}{}^2T_{day})$, where R_E is the earth's radius, R_{SE} is the earth-sun distance, T_{year} is one year, and

T_{day} is one day. Compute the value of this expression, and note that $L_{spin}/L_{orb} \ll 1$.

9.8 •• We have said that a classical picture of the electron as a spinning ball of matter is unsatisfactory. To illustrate this, consider the following: Modern measurements show that the electron's radius is certainly less than 10^{-18} m. Write an expression for the angular momentum of a uniform spinning ball of mass m_e, radius r, and equatorial speed v. By equating this to the observed spin $\sqrt{3}\hbar/2$, find the minimum possible value of v. What is v/c?

SECTION 9.3 (Magnetic Moments)

9.9 • A current of 0.4 A flows around a single circular loop of radius 1 cm. **(a)** What is the resulting magnetic moment, μ? **(b)** If the loop is placed in a magnetic field $B = 1.5$ T, with $\boldsymbol{\mu}$ perpendicular to **B**, what is the torque on the loop? **(c)** What is the difference in energy between the cases that $\boldsymbol{\mu}$ is parallel to **B** and antiparallel?

9.10 • The SI units of magnetic moment μ are ampere · meter^2. According to (9.13), the magnetic moment of an orbiting electron is

$$\boldsymbol{\mu} = -\frac{e}{2m_e}\mathbf{L} \qquad (9.35)$$

Verify that this has the correct units.

9.11 • A typical atomic magnetic moment is of order 10^{-23} A · m^2. Assuming that this is the result of a current i circulating around a single circular loop of radius 0.1 nm (a typical atomic radius), how big is i?

9.12 •• The energy of a magnetic moment $\boldsymbol{\mu}$ in a magnetic field B pointing along the z axis is $-\mu_z B$. For an electron in orbit around a proton, μ_z is given by (9.35) as $-(e/2m_e)L_z$. If $B = 10$ T (a large field by the standards of most laboratories) and if the electron is in a p state with $L_z = \hbar$, what is the magnetic energy due to the orbital magnetic moment (in joules and in eV)?

SECTION 9.4 (The Zeeman Effect)

9.13 • **(a)** Using the known SI values of e, $\hbar$, and m_e, find the SI value of the Bohr magneton $\mu_B = e\hbar/2m_e$. **(b)** Given that the units ampere · meter^2 are the same thing as joule/tesla, find μ_B in eV/tesla.

9.14 •• **(a)** Verify that the Bohr magneton $\mu_B = e\hbar/2m_e$ has the units of magnetic moment: namely, ampere · meter^2. **(b)** Verify that the units ampere · meter^2 are the same thing as joule/tesla. (Remember that the tesla is the unit of B field, defined by the Lorentz-force equation $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.)

9.15 •• A helium atom is in an energy level with one electron occupying an s state ($l = 0$) and the other an f state ($l = 3$). The two electron spins are antiparallel so that the spin magnetic moments cancel. The atom is placed in a magnetic field $B = 0.8$ T. **(a)** Sketch the resulting splitting of the original energy level. **(b)** What is the energy difference between adjacent levels of the resulting multiplet?

9.16 •• Imagine a hydrogen atom in which the electron has no spin [so that the only magnetic moment is the orbital magnetic moment given by (9.35)]. The atom is placed in a magnetic field $B = 1.5$ T along the z axis. **(a)** Describe the effect of the B field on the $1s$ and $2p$ states of the hypothetical atom. Sketch the energy levels. **(b)** When $B = 0$, there is a single spectral line corresponding to the $2p \to 1s$ transition. How many lines does this become when B is switched on? **(c)** What is the fractional separation $\Delta f/f_0$ between adjacent lines?

9.17 ••• Consider two levels of the helium atom in both of which the spins are antiparallel and one electron is in an s state ($l = 0$). In the higher level the second electron occupies a d state ($l = 2$), and in the lower level it occupies a p state ($l = 1$). **(a)** Sketch the splitting of both levels resulting from a magnetic field along the z axis. **(b)** Imagine a transition from one of the d states, with $L_z = m_i\hbar$, to one of the p states, with $L_z = m_f\hbar$. Since m_i can be 2, 1, 0, −1, or −2 and m_f can be 1, 0, or −1, there are 5 × 3 or 15 distinct conceivable transitions. How many different photon energies would these 15 transitions produce? **(c)** Not all of these 15 transitions occur. In fact, it is found that the only transitions observed are those for which

$$(m_f - m_i) = 1, \quad \text{or } 0, \quad \text{or } -1 \qquad (9.36)$$

(A restriction like this on the transitions that take place is called a *selection rule*, as we discuss in Chapter 11.) Prove that because of the restriction (9.36), there are only *three* distinct photon energies produced in all possible transitions. (This means that the normal Zeeman effect always produces just three spectral lines, however large the angular momenta involved.)

SECTION 9.5 (Spin Magnetic Moments)

9.18 • The electron's total magnetic moment $\boldsymbol{\mu}$ is given by (9.25). **(a)** What are the possible values of μ_z for an electron with $l = 0$? **(b)** Compare these with the values of μ_z for a hypothetical spinless electron with $l = 1$.

SECTION 9.6 (The Anomalous Zeeman Effect)

9.19 • Consider a hydrogen atom in its ground level, placed in a magnetic field of 0.7 T along the z axis. **(a)** What is the energy difference between the spin-up and spin-down states? **(b)** An experimenter wants to excite the atom from the lower to the upper state by sending in photons of the appropriate energy. What energy is this? What is the wavelength? What kind of radiation is this? (Visible? UV? etc.)

9.20 •• Consider a hydrogen atom in the $3d$ state with $L_z = 2\hbar$ and $S_z = \frac{1}{2}\hbar$. How much does its energy change if it is placed in a magnetic field $B = 0.6$ T along the z axis? [*Hint:* The total magnetic moment is given by (9.25), and the energy shift is $\Delta E = -\boldsymbol{\mu}_{tot} \cdot \mathbf{B}$.]

SECTION 9.7 (Fine Structure*)

9.21 •• The fine structure of an atomic spectrum results from the magnetic field "seen" by an orbiting electron. In this question you will make a semiclassical estimate of the B field seen by a $2p$ electron in hydrogen. The B field at the center of a circular current loop, i, of radius r is known to be $B = \mu_0 i/2r$. **(a)** Treating the electron and proton as classical particles in circular orbits (each as seen by the other), show that the B field seen by the electron is

$$B = \frac{\mu_0}{4\pi}\frac{eL}{m_e r^3} \tag{9.37}$$

where L is the electron's orbital angular momentum ($L = m_e vr$ for a circular orbit). Remember that the current produced by the orbiting proton is $i = ev/2\pi r$, where v is the speed of the proton as seen by the electron (or vice versa). **(b)** For a rough estimate, you can give L and r their values for the $n = 2$ orbit of the Bohr model, $L = 2\hbar$ and $r = 4a_B$. Show that this gives $B \approx 0.39$ T and hence that the separation, $2\mu_B B$, of the two $2p$ levels is about 4.5×10^{-5} eV.

It should be clear that this semiclassical calculation is only a rough estimate. You have used the Bohr values for L and r. If, for example, you had used the quantum value $L = \sqrt{2}\hbar$, this would have changed your answer by a factor of $\sqrt{2}$. There is another very important reason that the argument used here is only roughly correct: The electron's rest frame is noninertial (since it is accelerated) and a careful analysis by the British physicist L. H. Thomas showed that the energy separation calculated here should include an additional factor of $\frac{1}{2}$. That our answer, 4.5×10^{-5} eV, is correct to two significant figures is just a lucky accident.

9.22 •• **(a)** Use Eq. (9.37) (with the Bohr values $L = 2\hbar$ and $r = 4a_B$) to show that the fine-structure separation $\Delta E_{FS} = 2\mu_B B$ of the two $2p$ levels of hydrogen can be written as

$$\Delta E_{FS} = \frac{m_e(ke^2)^4}{32\hbar^4 c^2} \tag{9.38}$$

[*Hint:* Since $\mu_0\varepsilon_0 = 1/c^2$ and $k = 1/4\pi\varepsilon_0$, you can replace μ_0 by $\mu_0 = 4\pi k/c^2$.]

(b) Show that you can rewrite (9.38) as

$$\Delta E_{FS} = \frac{\alpha^2 E_R}{16} \tag{9.39}$$

where α is the dimensionless *fine-structure constant*

$$\alpha = \frac{ke^2}{\hbar c} \tag{9.40}$$

(c) Show that $\alpha \approx 1/137$, which, together with (9.39), shows that fine structure is indeed a small effect.

9.23 •• In Problem 5.10 it was shown that the speed of an electron in a Bohr orbit is of order $v \approx \alpha c$, where α is the fine-structure constant (9.40). Using this value, you can estimate the importance of relativistic corrections to the hydrogen energies, as follows: **(a)** Write down the correct relativistic expression for the electron's kinetic energy, and use the binomial series (Appendix B) to show that

$$K \approx \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2}$$

provided that $v \ll c$ (which is certainly true if $v \approx \alpha c$). Thus the relativistic correction ΔE_{rel} to the energy is about $3mv^4/8c^2$. **(b)** Substitute $v \approx \alpha c$ and show that this gives

$$\Delta E_{rel} \approx \frac{3\alpha^2 E_R}{4}$$

Comparing this with (9.39), we see that relativistic corrections to the hydrogen energy are of the same order as the spin-orbit correction. Therefore, a correct treatment of fine structure must include both effects.

SECTION 9.8 (Magnetic Resonance Imaging*)

9.24 • Just as the electron magnetic moment is given approximately by the Bohr magneton (9.17), the nuclear magnetic moment is conveniently expressed in terms of the **nuclear magneton**,

$$\mu_N = \frac{e\hbar}{2m_p} \tag{9.41}$$

where m_p is the proton mass. **(a)** Find the value of μ_N in eV/T. Compare this with the Bohr magneton. **(b)** The proton's magnetic moment is found to be $2.793\mu_N$. (This common — and somewhat ambiguous — statement means that the two values of μ_z are $\pm 2.793\,\mu_N$.) Find the energy-level separation for a proton in a magnetic field $B = 1$ T.

9.25 • Use the results of the previous problem to show that for a proton in a $B = 1$-tesla field, the energy-level splitting corresponds to a frequency of 42.6 MHz.

9.26 • According to statistical mechanics (Ch. 15), if a particle is in thermal equilibrium at temperature T and it can exist in one of two quantum states with energies E_1 and E_2, then the probabilities $P(E_1)$ and $P(E_2)$ that the particle will occupy each of the two levels are related by

$$\frac{P(E_2)}{P(E_1)} = \frac{e^{-E_2/kT}}{e^{-E_1/kT}}$$

where k is Boltzmann's constant. For a proton in a magnetic field $B = 1$ T at temperature $T = 300$ K, compute the ratio P(higher energy)/P(lower energy) using the result of Problem 9.24. Note how close to 1 this result is. Because of thermal agitation, a collection of protons is only very weakly magnetized when placed in a strong magnetic field; that is, there is only a very slight excess of moments aligned with the field over those anti-aligned. Consequently, the proton radio signal in MRI is exceedingly weak, and several minutes of signal averaging are required to produce a high-quality image.

9.27 ••• *Semi-classical model of spin in a magnetic field.* A large collection of proton spins in a magnetic field acts, in many ways, like a classical magnetic moment $\boldsymbol{\mu}$. Given that the torque on a moment in a field is given by $\boldsymbol{\Gamma} = \boldsymbol{\mu} \times \mathbf{B}$, and that the classical angular momentum $\mathbf{L}$ is related to the moment $\boldsymbol{\mu}$ by $\boldsymbol{\mu} = \gamma\mathbf{L}$, and finally that $\boldsymbol{\Gamma} = d\mathbf{L}/dt$, show that a moment $\boldsymbol{\mu}$ in a field $\mathbf{B}$ will *precess* about the field direction with an angular frequency given by $\omega = \gamma B$.