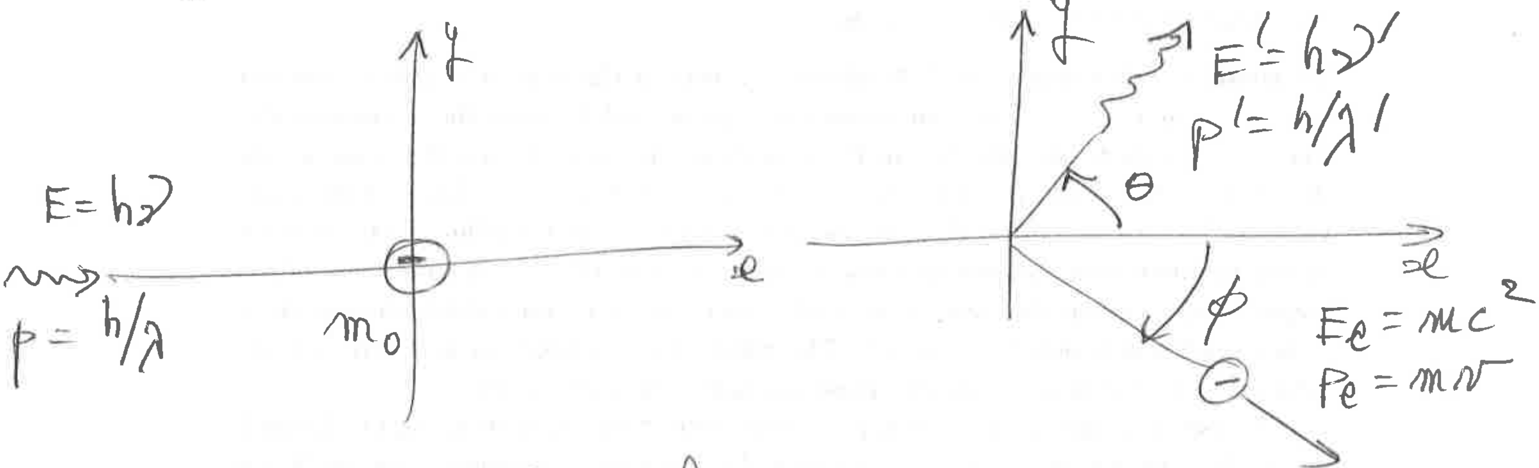
Compton Effect

Classically: If electromagnetic radiation is diffused after interaction with a charged particle, the frequency of the diffused light is the same as the incident one.

Quantum-mechanically: Compton revisited the problem using quantum interpretation of light in 1922.



Compton has shown

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0 c}(1-\cos\theta)$$

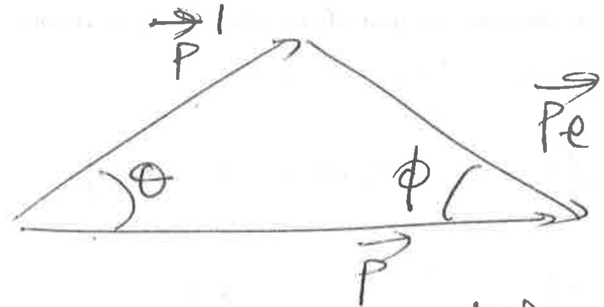
angle between light beam before and after scattering

The quantity $\frac{h}{m_0 c}$ is called the Compton wavelength

For an electron $\frac{h}{m_0 c} = 0.0243\,\mathring{A}$.

$\frac{h}{m_0 c}$ does not depend on the energy of the incident photon.

Derivation of Compton formula



photon has an energy $E = h\nu = hc/\lambda$

momentum $p = h/\lambda$

Conservation of energy

$$\frac{hc}{\lambda} + m_0c^2 = \frac{hc}{\lambda'} + mc^2$$

$$\rightarrow mc^2 = m_0c^2 + hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$= m_0c^2 + hc\left(\frac{\lambda' - \lambda}{\lambda\lambda'}\right)$$

Squaring on both sides

$$(mc^2)^2 = \frac{h^2c^2}{\lambda^2\lambda'^2}(\lambda^2 + \lambda'^2) - \frac{2h^2c^2}{\lambda\lambda'}$$

$$+ \frac{2hm_0c^3}{\lambda\lambda'}(\lambda' - \lambda) + (m_0c^2)^2 \quad (1)$$

Conservation of momentum

$$\vec{p_e} = \vec{p} - \vec{p}\,'$$

$$\vec{P_e}\cdot\vec{P_e} = p^2 + p'^2 - 2\vec{p}\cdot\vec{p}'$$

$$= \frac{h^2}{\lambda^2\lambda'^2}\left(\lambda'^2 + \lambda^2 - 2\lambda\lambda'\cos\theta\right) \qquad (2)$$

But for relativistic particles

$$(mc^2)^2 = (P_e c)^2 + (m_0 c^2)^2$$

$$\cancel{\frac{h^2c^2}{\lambda^2\lambda'^2}(\lambda^2+\lambda'^2)} - \frac{2h^2c^2}{\lambda\lambda'} + \frac{2hm_0c^3}{\lambda\lambda'}(\lambda'-\lambda) + \cancel{(m_0c^2)^2}$$

$$= \cancel{\frac{h^2c^2}{\lambda^2\lambda'^2}(\lambda'^2+\lambda^2} - 2\lambda\lambda'\cos\theta) + \cancel{(m_0c^2)^2}$$

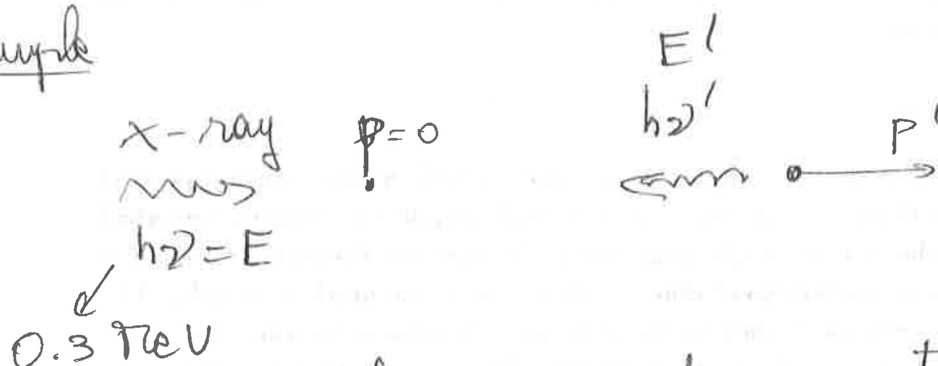
$$\frac{2hm_0c^3}{\lambda\lambda'}(\lambda'-\lambda) = \frac{2h^2c^2}{\lambda\lambda'}\left[1-\cos\theta\right]$$

$$\boxed{\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0c}(1-\cos\theta)}$$

!

Compton has checked the validity of his theory by looking at the diffraction of x-ray ($\lambda = 0.7\,Å$) by graphite. The energy of the x-ray is $1.8\,10^4$ eV. is much larger than the energy of bound electrons in graphite. These bound electrons can be assumed as free electrons.

Example



Use conservation of energy and momentum and calculate the recoil velocity of the electron.

$$\overset{\text{x-ray}}{E} + m_0c^2 = E' + \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$0.3\,\text{MeV} + 0.511\,\text{MeV} = E' + \frac{0.511\,\text{MeV}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

For photon $\quad p = \frac{h}{\lambda} = \frac{h}{\frac{hc}{E}} = \frac{E}{c}$

Conservation of momentum

$$\frac{E}{c} + 0 = -\frac{E'}{c} + \frac{m_0 v}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

So $$\frac{0.3\ \text{MeV}}{c} = -\frac{E'}{c} + \frac{0.511\ \text{MeV}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\ \frac{v}{c}$$

We must solve simultaneously

(1) $$0.3\ \text{MeV} + 0.511\ \text{MeV} = E' + \frac{0.511\ \text{MeV}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

(2) $$\frac{0.3\ \text{MeV}}{c} = -\frac{E'}{c} + \frac{0.511\ \text{MeV}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\left(\frac{v}{c^2}\right)$$

$\times(c) \rightarrow$

(2') $$0.3\ \text{MeV} = -E' + \frac{0.511\ \text{MeV}}{\sqrt{1-x^2}}\ x \qquad x = \frac{v}{c}$$

Adding (1) & (2')

$$1.111\ \text{MeV} = 0.511\ \text{MeV}\ \frac{(1+x)}{\sqrt{1-x^2}}$$

$$\frac{1.111}{0.511}\sqrt{1-x^2} = (1+x)$$

$$a\sqrt{1-x^2} = (1+x) \quad \text{where } a = \frac{1.111}{0.511}$$

Squaring both sides

$$a^2(1-x^2) = 1 + x^2 + 2x$$

$$x^2(1+a^2) + 2x + (1-a^2) = 0$$

$$x = \frac{-2 \pm \sqrt{(1+a^2)^2 - 4(1-a^4)}}{4} \rightarrow x = 0.65$$

is only acceptable solution

Derive an analytical expression for the kinetic energy of the diffused electron in Compton scattering as a function of the angle ϕ. (6)

$$h\nu + m_0c^2 = h\nu' + K_e + m_0 c^2$$

But $h\nu = pc$

$$pc = p'c + K_e \qquad (*)$$

$$p'^2 = \vec{p}'\cdot\vec{p}' = p^2 + p_e^2 - 2pp_e\cos\phi \qquad (**)$$

Also $$p_e^2 = \frac{E_e^2 - (m_0c^2)^2}{c^2}$$

$$= \frac{1}{c^2}\left[(m_0c^2+K_e)^2 - (m_0c^2)^2\right]$$

$$p_e^2 = \frac{1}{c^2}\left(K_e^2 + 2K_e m_0 c^2\right) \qquad (***)$$

$(***)$ and $(*)$ in $(**)$

$$\left(p + \frac{K_e}{c}\right)^2 = p^2 + \frac{1}{c^2}\left(K_e^2 + 2K_e m_0 c^2\right) - 2pp_e\cos\phi$$

$$\cancel{p^2} + 2\frac{pK_e}{c} + \cancel{\frac{K_e^2}{c^2}} = \cancel{p^2} + \cancel{\frac{K_e^2}{c^2}} + 2K_e m_0 - 2pp_e\cos\phi$$

$$K_e\left(m_0 + \frac{p}{c}\right) = pp_e\cos\phi$$

Squaring and using $(***)$

$$K_e^2\left(m_0 + \frac{p}{c}\right)^2 = \frac{p^2}{c^2}\left(K_e^2 + 2K_e m_0 c^2\right)\cos^2\phi$$

But $p = \frac{h\nu}{c}$

$$K_e^2\left(m_0 + \frac{h\nu}{m_0c^2}\right)^2 = \frac{p^2}{c^2}\left(K_e^2 + 2K_e m_0 c^2\right)\cos^2\phi$$

$$K_e^2\left[\left(m_0 + \frac{p}{c}\right)^2 - \left(\frac{p}{c}\right)^2\cos^2\phi\right] = \frac{2K_e m_0 c^2}{c^2}p^2\cos^2\phi$$

$$K_e^2\left[m_0^2 + 2\frac{m_0}{c}\frac{h\nu}{c} + \left(\frac{p}{c}\right)^2\left[1-\cos^2\phi\right]\right]$$
$$= 2K_e m_0\left(\frac{h\nu}{c}\right)^2\cos^2\phi$$

$$K_e^{\cancel{2}}\left[1 + 2\frac{h\nu}{m_0c^2} + \frac{(h\nu)^2}{m_0^2c^4}\left(1-\cos^2\phi\right)\right]$$
$$= \frac{2\cancel{K_e}}{m_0}\left(\frac{h\nu}{c}\right)^2\cos^2\phi$$

$$K_e = h\nu\left[\frac{2\left(\frac{h\nu}{m_0c^2}\right)\cos^2\phi}{\left(1+\frac{h\nu}{m_0c^2}\right)^2 - \left(\frac{h\nu}{m_0c^2}\right)^2\cos^2\phi}\right]$$

$$y = \frac{K_e}{m_0c^2} = \underbrace{\left(\frac{h\nu}{m_0c^2}\right)}_{x}\left[\frac{2\left(\frac{h\nu}{m_0c^2}\right)\cos^2\phi}{1+\left(\frac{h\nu}{m_0c^2}\right)^2 - \left(\frac{h\nu}{m_0c^2}\right)^2\cos^2\phi}\right]$$

plot y versus ϕ ~~cos~~ for different x.