

Chapter 11

Atomic Transitions and Radiation

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*These sections could be omitted without significant loss of continuity.

11.1 Introduction

In this chapter we discuss the emission and absorption of radiation by atoms making transitions between different energy levels. Our general observations will also apply to a wide variety of other quantum systems, including ions, molecules, and nuclei. We begin with a classical description of radiation and apply it to a classical model of the atom with electrons in circular orbits. In doing so, we discover some useful concepts and see how the classical view of an atom fails to agree with observation. Next we will see how radiation striking a quantum atom can cause transitions between quantum states. This process accounts for the emission or absorption of a photon by an atom exposed to external radiation, and also for the spontaneous emission of a photon by an isolated atom in an excited state. Armed with this understanding, we conclude the chapter by describing one of the most exciting applications of quantum theory, the laser.

11.2 Radiation by Classical Charges

An accelerating charge produces electric and magnetic fields that change with time. Faraday's law of induction tells us that a changing magnetic field induces an electric field, and, as first proposed by Maxwell, a changing electric field induces a magnetic field. This suggests that an accelerating charge can bring about an intertwined process in which changing electric and magnetic fields continually induce one another, with the fluctuations in these fields moving

away from the charge as electromagnetic radiation. This suggestion proves to be correct: All of the many forms of electromagnetic radiation around us — light from a lightbulb or the sun, radio waves from our favorite radio station, X-rays in our doctor's office — are produced by accelerating charges.

In Maxwell's theory of electromagnetic radiation these ideas are made precise by showing that the laws of magnetic and electric induction can be combined to give a wave equation for both the magnetic and electric fields. The wave speed c for disturbances of either field is given by the combination of electromagnetic constants

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s}$$

which equals the observed speed of light. This suggested to Maxwell, what has since been amply verified, that light consists of electromagnetic waves in which electric and magnetic fields travel together.

To get some picture of how an accelerating charge produces radiation, let us examine the electric field of a single charge that is abruptly put into motion. In Fig. 11.1(a) we show the initially static charge and its radial field. When the charge is put into motion, the news of its changed position moves outward at the speed of light, so distant portions of the field lines are offset from near portions. The transition zone between the near and far fields necessarily contains a transverse component, as shown in Fig. 11.1(b) and (c).^{*} While the radial component of the electric field falls like $1/r^2$ it can be shown that the transverse component falls like $1/r$. Consequently, at large distances it is the transverse component that dominates and carries radiated energy away from the charge.

The total power P radiated by any single charge q (moving nonrelativistically[†]) can be shown to be

$$P = \frac{2kq^2a^2}{3c^3} \quad (11.1)$$

where a is the charge's acceleration. This formula accurately describes the power radiated by any macroscopic system of moving charges. For example, in TV or radio broadcasting, electric charges are made to oscillate inside the rods of an antenna, and the resulting radiated power is given by (11.1). (See Problem 11.2.) Notice that the power (11.1) depends on the acceleration a . Thus a charge moving at constant velocity does not radiate.[‡] We should also mention that with an assembly of *many* accelerating charges, the fields produced by the different charges can sometimes interfere destructively, with no net radiated power. For example, consider a uniform ring of charge rotating at a constant rate; this amounts to a steady current loop and does not radiate any power.

^{*}Fig. 11.1 shows the field lines for a charge with speed much less than c . The pattern for relativistic speeds is more complicated. For a beautiful discussion of the fields from accelerated charges, see Edward Purcell, *Electricity and Magnetism* (McGraw-Hill, 1964), Chapter 5.

[†]To see how (11.1) is modified when the motion is relativistic, see Problem 11.7.

[‡]To see why no radiation occurs when \mathbf{v} is constant, consider the inertial frame in which $\mathbf{v} = 0$. In this frame the magnetic field is zero and the electric field is constant, so no radiation occurs. It follows from the principle of relativity that no radiation occurs in any frame.

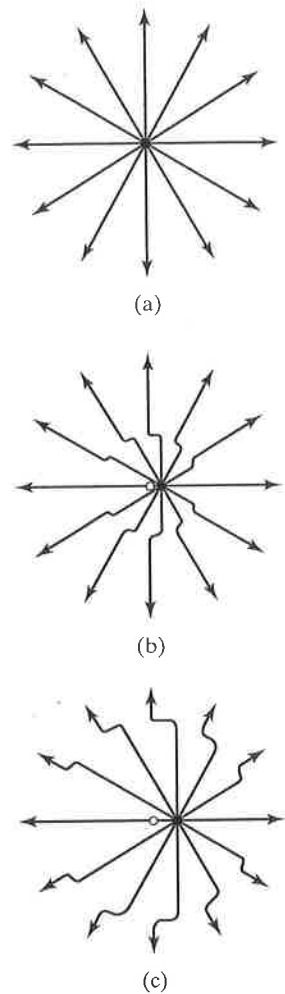


FIGURE 11.1

(a) Electric field lines from a static charge are radial. (b) When the charge is given an abrupt kick to the right, changes in its electric field propagate outward at speed c ; distant portions of the field still point outward from the original (open circle) position. (c) The transverse disturbance linking near and far fields continues to move radially outward as the charge coasts forward.

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The formula (11.1) and the classical theory from which it derives can sometimes be applied to microscopic systems as well. For example, when an electron in an X-ray tube collides with the anode, it undergoes rapid deceleration. The classical theory correctly predicts that this deceleration will produce the X-ray radiation, known as bremsstrahlung, or braking radiation (Section 4.4). Since the classical theory takes no account of the quantization of radiation, it cannot, of course, be correct in every detail. For example, classical theory predicts that some bremsstrahlung X-rays will be produced at all frequencies. In reality, each photon carries energy hf , and no photons can be produced for which hf exceeds the kinetic energy K of the incident electron. Therefore, the spectrum of emitted X-rays is cut off abruptly at $f = K/h$ as predicted by the Duane–Hunt law of Section 4.5.

When the formula (11.1) was applied to a classical atom, difficulties arose almost immediately, and it was this problem that suggested to Bohr that classical physics needed modification, as described in Section 5.4. The problem in question is that an electron orbiting in a classical atom is necessarily accelerating. Therefore, according to (11.1), it must be radiating electromagnetic energy. In the following example we see that the rate of radiation predicted by (11.1) is so large that a classical atom would collapse completely in a time on the order of 10^{-11} seconds.

Example 11.1

Find the power that would be radiated by a classical electron in the $n = 1$ Bohr orbit of a hydrogen atom.

The orbiting electron has a centripetal acceleration $a = v^2/r$ and must therefore radiate. Perhaps the simplest way to find the acceleration is to note that it is given by Newton's second law as $a = F/m$, with $F = ke^2/r^2$. Substituting into (11.1), we find that

$$P = \frac{2ke^2}{3c^3} \left(\frac{ke^2}{mr^2} \right)^2 = \frac{2(ke^2)^3 c}{3(mc^2)^2 r^4} \quad (11.2)$$

or, putting $r = a_B = 5.3 \times 10^{-2}$ nm,

$$\begin{aligned} P &= \frac{2 \times (1.44 \text{ eV} \cdot \text{nm})^3 \times (3 \times 10^{17} \text{ nm/s})}{3 \times (5.1 \times 10^5 \text{ eV})^2 \times (5.3 \times 10^{-2} \text{ nm})^4} \\ &= 2.9 \times 10^{11} \text{ eV/s} \end{aligned} \quad (11.3)$$

where we have again used the useful combination $ke^2 = 1.44 \text{ eV} \cdot \text{nm}$. Since typical energies of electrons in atoms are a few eV, the rate (11.3) is an extremely rapid energy loss. (For more details, see Problems 11.6 and 11.15.)

The result (11.3) shows that the orbit of an electron with an energy of a few eV will change radically in a time of order 10^{-11} seconds. More precisely, as the electron loses energy, its orbital radius will shrink, and once r reaches zero, the atom will have collapsed completely. As you can check in Problem 11.15, the time τ for a hydrogen atom in its ground state to collapse in this way would be

$$\tau = 1.6 \times 10^{-11} \text{ s} \quad (11.4)$$

This dramatic instability of the classical atom was what led Bohr to postulate the existence of quantized orbits to which the classical laws of radiation did not apply. As we saw in Chapter 7 (and review in Section 11.3), the Schrödinger theory avoids this problem since it predicts that atoms have stationary charge distributions.

11.3 Stationary States and Transitions

In Section 11.2 we saw that the radiation from a classical atom would lead to a rapid collapse of the electrons' orbits. We now turn to a proper quantum treatment of the atomic electrons and will see how this disastrous collapse is avoided. As we discussed in Section 7.3, the wave function $\Psi(\mathbf{r}, t)$ of an electron generally depends on time as well as the spatial coordinates. In particular, suppose that at time $t = 0$ the wave function is a stationary state $\Psi(\mathbf{r}, 0) = \psi_n(\mathbf{r})$ with energy E_n ; that is, ψ_n satisfies the time-independent Schrödinger equation with $E = E_n$. Then at any later time t the complete wave function $\Psi(\mathbf{r}, t)$ is a product of $\psi_n(\mathbf{r})$ and the oscillatory phase factor $e^{-iE_n t/\hbar}$, as shown here:

$$\Psi(\mathbf{r}, t) = \psi_n(\mathbf{r})e^{-iE_n t/\hbar} \quad (11.5)$$

as in Eq. (7.11). In the problems considered so far, we have been able to restrict attention to the spatial wave function $\psi_n(\mathbf{r})$. However, since transitions are a distinctly time-dependent phenomenon, we must now examine the full time-dependent wave function $\Psi(\mathbf{r}, t)$.

As we discussed in Chapter 6, the probability density P of an electron is given by the square of the absolute value of its wave function

$$P(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|^2 \quad (11.6)$$

which for the wave function (11.5) becomes

$$P(\mathbf{r}, t) = |\psi_n(\mathbf{r})|^2 \times |e^{-iE_n t/\hbar}|^2 = |\psi_n(\mathbf{r})|^2 \quad (11.7)$$

since $|e^{-i\theta}| = 1$. (See Sec. 7.3.) We see in (11.7) that the probability density (and hence the charge density) of an electron in the state (11.5) does not depend on time. It is for this reason that these states are called *stationary states*. Their charge distributions are stationary and hence do not radiate.*

The simple Schrödinger theory outlined here successfully explains the existence of stable atomic states. Unfortunately, it goes too far, since it predicts that all states of definite energy should be perfectly stable. Thus, our task now is to explain how transitions *can* occur, in which an atom or other quantum system can move from one energy level to another. We will describe the theory of transitions in the next two sections, but let us first sketch the main features here.

If the initial wave function $\Psi(\mathbf{r}, 0) = \psi_n(\mathbf{r})$ really is a solution of the time-independent Schrödinger equation, including the correct potential energy U of the isolated atom, then the full time-dependent wave function is given

*You might object that in a stationary state with $l \neq 0$ the charge distribution may be stationary, but the electron is surely circulating since it has nonzero orbital angular momentum. This is correct, but this situation corresponds to a steady current loop and still does not radiate.

by (11.5) and the atom remains locked in the same state for all time. For transitions to occur, there has to be some outside influence that adds an additional term W to the potential energy. For example, we might switch on an electric field \mathcal{E} in the x direction. In this case the electron would acquire an additional potential energy

$$W = e\mathcal{E}x \quad (11.8)$$

This extra potential energy is called a **perturbation**, since it perturbs the system and causes transitions. Perturbations can come about in several ways; for example, a pulse of radiation directed at an atom can produce a brief oscillatory electric field causing a potential energy of the form (11.8). The passage of a second atom close by the atom of interest can change the potential energy of the latter and cause transitions. Finally, as we will argue in Section 11.7, even an atom that is perfectly isolated is subject to tiny electric and magnetic fields called vacuum fluctuations or zero-point fields, and it is these vacuum fluctuations that cause an excited but isolated atom to emit a photon and drop to a lower energy level.

It is easy to understand in a general way how a perturbation like (11.8) can cause transitions. The wave function ψ_n is a solution of the Schrödinger equation, including just the potential energy U of the isolated atom. It is *not* a solution of the Schrödinger equation with the full potential energy $U + W$ and is not, therefore, a stationary state of the actual problem including the perturbation. Therefore a system that starts out in the state ψ_n can change to some other state ψ_m .

The theory of transitions caused by a perturbation is called **time-dependent perturbation theory**. As we will describe in Sec. 11.5, when a system starts out at time $t = 0$ in a state $\Psi(\mathbf{r}, 0) = \psi_n(\mathbf{r})$ and is subjected to a perturbation W , there is a definite probability $P(n \rightarrow m)$ that a subsequent measurement will find the system in a different state $\psi_m(\mathbf{r})$. In some simple situations, we will see how this probability can actually be calculated, but the main purpose of the next two sections is just to see how quantum mechanics can account for the occurrence of transitions. If you feel happy to accept the claims just made, then you can skip these two sections without significant loss of continuity.

11.4 More Quantum Formalism*

*This section can be omitted without significant loss of continuity.

Before we describe the quantum-mechanical theory of transitions, it is useful to introduce some changes of notation and a couple of new ideas. These are developments that you will certainly meet if you take another course in quantum mechanics, and now is a good opportunity to introduce them.

The Hamiltonian Operator

Our first development concerns the way that we write the time-independent Schrödinger equation,

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2} [U(x) - E]\psi \quad (11.9)$$

(For simplicity, we'll consider a one-dimensional system for now.) The form (11.9) is convenient for study as a differential equation, but it is not the form in which the equation is normally written in more advanced work, where it is usually reorganized to read

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + U(x) \right] \psi = E\psi \quad (11.10)$$

The quantity in the square brackets on the left is a **differential operator**, or just **operator**: "Operating" on a function ψ , it gives the function $-\frac{\hbar^2}{2m} \psi'' + U\psi$. This operator is so important, it is given a name, the **Hamiltonian operator**, or just **Hamiltonian**,* and is denoted H

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + U(x) \quad (11.11)$$

The Hamiltonian is intimately connected to the energy of the system. For example, if ψ is the plane wave $\psi = e^{ikx}$,

$$H\psi = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + U \right] e^{ikx} = \left[\frac{\hbar^2 k^2}{2m} + U \right] \psi = [K + U]\psi \quad (11.12)$$

In other words, the operator H acting on a plane wave gives $(K + U)$ times the plane wave. More important, the time-independent Schrödinger equation can now be written as

$$H\psi = E\psi \quad (11.13)$$

from which we see that for any wave function ψ that has a definite energy E , the differential operator H acting on ψ gives simply E times ψ . In this sense, the Hamiltonian operator of any system determines the allowed energies of the system.

One of the great advantages of this new notation is this: If we were to consider a three-dimensional particle, the Schrödinger equation in the form corresponding to (11.10) would become

$$\left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + U(x) \right] \psi = E\psi \quad (11.14)$$

But if we agree to define the Hamiltonian for a three-dimensional particle as

$$H = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + U(x)$$

the three-dimensional Schrödinger equation takes the same simple form, $H\psi = E\psi$, as in one dimension. No matter how complicated our system, we

*The origin of the name is this: In classical mechanics, the Hamiltonian function is a function that gives the energy of a system (named after the Irish mathematician William Rowan Hamilton). As we will argue directly, the Hamiltonian operator plays an analogous role in quantum theory.

can always define the Hamiltonian H so that the Schrödinger equation takes this simple form, $H\psi = E\psi$. Of course, the operator H may become very complicated, but the great advantage of our new notation is that we can discuss the Schrödinger equation for *all conceivable systems* at the same time in the simple form $H\psi = E\psi$. For those readers who haven't seen this kind of equation before, we should emphasize the form of the equation:

$$(\text{differential operator}) \times \text{function} = \text{number} \times (\text{same function})$$

This type of equation is called an **eigenvalue equation**, and the values of E for which it has solutions (namely, the allowed energies) are called **eigenvalues**.

The equation that determines the time development of an arbitrary quantum state (of definite energy or otherwise) is the time-dependent Schrödinger equation, which we introduced in Section 7.11.* For a one-dimensional particle, this is

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x) \right] \Psi(x, t) \quad (11.15)$$

which we can immediately rewrite in terms of the Hamiltonian operator as

$$i\hbar \frac{\partial}{\partial t} \Psi = H\Psi \quad (11.16)$$

and in this form, the equation applies to *any* system (although, of course, the details of the operator H vary from system to system). This is the equation we will use to study how atoms make their transitions from one energy level to another.

A Useful Approximation

It is very seldom that one can solve the time-dependent Schrödinger equation exactly, and physicists have developed many schemes for finding approximate solutions. In this chapter we will use just one simple scheme, which lets us find Ψ at a time $t + \Delta t$ in terms of its value at time t , at least if Δt is small. This scheme is based on the familiar approximation

$$f(t + \Delta t) \approx f(t) + \Delta t \frac{df}{dt}$$

which follows from the definition of the derivative. In the case of the wave function Ψ , the time derivative $\partial\Psi/\partial t$ is given by the time-dependent Schrödinger equation (11.16) and the corresponding approximation becomes

$$\Psi(t + \Delta t) \approx \Psi(t) + \frac{\Delta t}{i\hbar} H\Psi(t) \quad (11.17)$$

Notice that Ψ depends on other variables besides t [for a particle in one dimension $\Psi = \Psi(x, t)$ as in (11.15); for a particle in three dimensions, $\Psi = \Psi(x, y, z, t)$; and so on], but for now, it is convenient to omit these.

*If you skipped Section 7.11, you should return to read it now, before continuing with this and the next section.

The approximation (11.17) tells us what gets added to Ψ as the time advances by Δt . It is interesting to check this for the case that Ψ happens to be a stationary state: Suppose that at $t = 0$, $\Psi(0) = \psi_n$, a state with definite energy E_n , satisfying the time-independent Schrödinger equation $H\psi_n = E_n\psi_n$. According to (11.17), the wave function a short time later is

$$\Psi(\Delta t) \approx \Psi(0) + \frac{\Delta t}{i\hbar} H\psi_n = \psi_n - \frac{i\Delta t}{\hbar} E_n\psi_n = \left(1 - \frac{iE_n\Delta t}{\hbar}\right)\psi_n \quad (11.18)$$

That is, the new wave function at time Δt is the original function ψ_n multiplied by a (complex) number and represents the same physical state (as we knew it must, since it is a stationary state). In fact, the multiplicative factor that appears in the last expression of (11.18) is easily understood, since we know the exact solution of the time-dependent Schrödinger equation in this case: As in (11.5)

$$\Psi(t) = e^{-iE_n t/\hbar}\psi_n = \left(1 - \frac{iE_n t}{\hbar} - \frac{E_n^2 t^2}{2\hbar^2} + \dots\right)\psi_n \quad (11.19)$$

where the sum in parentheses is just the Taylor series for the exponential factor in the previous expression. Comparing (11.18) with (11.19), we see that the approximation (11.18) is just the first two terms of the Taylor series for the exact form of $\Psi(\Delta t)$ and is certainly a good approximation for small time intervals Δt .

Completeness of the Stationary-State Wave Functions

The stationary states of any quantum system are the solutions of the time-independent Schrödinger equation, $H\psi_n = E_n\psi_n$. It is a remarkable fact, though not one we can prove here, that *any* wave function ψ of the system (stationary state or not) can be expanded in terms of these stationary-state wave functions. That is, any wave function ψ can be written as

$$\psi(x) = \sum_n A_n \psi_n(x) \quad (11.20)$$

where the coefficients A_n are constant complex numbers, which depend on the particular function $\psi(x)$ we are expanding (and we have written the expansion for the case of a one-dimensional particle, though it is equally valid for any quantum system). A set of functions with this property, which can be used to expand any function, is often called a **complete set of functions**, or a **basis set**. Thus, we can rephrase our claim to say that, for any quantum system, the stationary-state wave functions are a complete set for the expansion of any wave function of the system. The significance of the expansion coefficients A_n is this: $|A_n|^2$ is the probability that a measurement of the energy will result in the value E_n .

Example 11.2

Recall the stationary-state wave functions for a particle of mass m in a one-dimensional rigid box (the infinite square well) of width a , and write down the expansion of an arbitrary wave function $\psi(x)$ in terms of these stationary-state functions. What are the expansion coefficients A_n if $\psi(x)$ is in fact the

ground-state wave function? What if $\psi(x)$ is the first excited state? What if $\psi(x)$ is a 50–50 mixture of the lowest two states?

The stationary-state wave functions for the rigid box are given in (7.60):

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \quad [n = 1, 2, 3, \dots] \quad (11.21)$$

So the expansion (11.20) reads

$$\psi(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \quad (11.22)$$

which you may recognize as the Fourier expansion of an arbitrary function in terms of sine functions, as sketched in Chapter 6. This example illustrates a common occurrence: For a particular quantum system (the infinite square well here), the completeness of the stationary-state functions is often derivable from some standard mathematical theorem (Fourier's expansion theorem in this case). In general, however, it is just a postulate of quantum theory that all systems have this property.

If the wave function $\psi(x)$ is known to be the ground state $\psi_1(x)$, then clearly

$$A_1 = 1, \quad \text{and} \quad A_2 = A_3 = \dots = 0$$

This shows that the probability of finding the energy $E = E_1$ is $|A_1|^2 = 1$, while the probability of finding any other value is 0. Similarly, if $\psi(x)$ is known to be the first excited state $\psi_2(x)$, then

$$A_1 = 0, \quad A_2 = 1, \quad \text{and} \quad A_3 = A_4 = \dots = 0$$

Finally, if $\psi(x)$ is known to be a 50–50 mixture of the two lowest states, the probabilities of finding $E = E_1$ or $E = E_2$ must each be a half, so $|A_1|^2 = |A_2|^2 = 1/2$; that is, both A_1 and A_2 must have absolute value $1/\sqrt{2}$. This does not quite fix the two coefficients, and indeed there are several distinct wave functions that can all be described as 50–50 mixes of the two lowest states. One example would have

$$A_1 = A_2 = 1/\sqrt{2}, \quad \text{and} \quad A_3 = A_4 = \dots = 0$$

but another would have $A_1 = -A_2 = 1/\sqrt{2}$ and so on.

Orthogonality of the Stationary-State Wave Functions

If we know the stationary-state wave functions $\psi_n(x)$ for some system and we are given an arbitrary function $\psi(x)$, then it is surprisingly easy to find the coefficients A_n in the expansion (11.20). This is because of a remarkable property of any set of stationary-state wave functions, called **orthogonality**: If

$\psi_n(x)$ (with $n = 1, 2, \dots$) are the stationary states of any quantum system, then it can be shown that*

$$\int_{-\infty}^{\infty} \psi_m(x)^* \psi_n(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \quad (11.23)$$

That this integral equals 1 if $m = n$ is just a statement that the wave functions are normalized (as we assume throughout this discussion). That it is zero if $m \neq n$ is a new result, often expressed by saying that ψ_m is **orthogonal** to ψ_n , since the integral in (11.23) is in some ways analogous to the scalar product $\mathbf{a} \cdot \mathbf{b}$ of two vectors \mathbf{a} and \mathbf{b} , and the scalar product is zero if the vectors are orthogonal. For a proof of the orthogonality property for one particular system, see Problem 11.17.

Finding the Expansion Coefficients

Armed with the orthogonality relation (11.23), it is a simple matter to find the coefficients A_n in the expansion (11.20) of any given function $\psi(x)$. Multiplying both sides of (11.20) by $\psi_m(x)^*$ and integrating, we find that

$$\int_{-\infty}^{\infty} \psi_m(x)^* \psi(x) dx = \sum_{n=0}^{\infty} A_n \int_{-\infty}^{\infty} \psi_m(x)^* \psi_n(x) dx$$

By (11.23), all of the integrals on the right are zero, except the one with $n = m$, which is equal to 1. Thus the whole infinite series collapses to a single term, equal to A_m , and we conclude that

$$A_m = \int_{-\infty}^{\infty} \psi_m(x)^* \psi(x) dx \quad (11.24)$$

If you did either of Problems 6.35 or 6.36, you will recognize the result (11.24) as the method used to find the coefficients in a Fourier series. We will use the result (11.24) in the next section to find the probabilities of transitions between different energy levels of a quantum system.

11.5 Transitions; Time-Dependent Perturbation Theory*

*Like Section 11.4, this section can be omitted without significant loss of continuity. However, you do need to have read 11.4 before reading 11.5.

We are now ready to see how an atom or other quantum system in a stationary state ψ_n can make a transition to another stationary state ψ_m . First, for reasons that will emerge directly, we will call the Hamiltonian operator for

*We state the result here for a particle in one dimension, but it extends in a natural way to any quantum system. For example, for a three-dimensional particle, we have only to replace the integral over x by an integral over $x, y,$ and z . We have stated the orthogonality property for the case that the energy levels are nondegenerate; if there are any degeneracies, the statement needs a little more care, though the practical consequences are the same.

the isolated system H_0 . For example, if our system is an electron in one dimension, then

$$H_0 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x)$$

If U is the exact potential energy of our isolated system and if the system is initially in the state ψ_n (which satisfies the time-independent Schrödinger equation $H_0\psi_n = E_n\psi_n$), we know that the full wave function for all time has the form (11.5),

$$\Psi(t) = \psi_n e^{-iE_n t/\hbar}$$

Therefore, the system remains in the same stationary state ψ_n for all time, and no transitions can occur. For transitions to occur, the true Hamiltonian H of the system must contain an additional term,

$$H = H_0 + W \quad (11.25)$$

At the end of Sec. 11.3, we mentioned several ways in which this extra perturbation W can come about. For the moment, let us suppose it arises from an electric field \mathcal{E} in the x direction, switched on at time 0, in which case the electron acquires the additional potential energy (11.8)

$$W = e\mathcal{E}x \quad (11.26)$$

The situation is now this: At time $t = 0$, our system is in the state $\Psi(0) = \psi_n$. The state ψ_n is a stationary state, but it is a stationary state of the "wrong" Hamiltonian, H_0 . The true Hamiltonian is $H = H_0 + W$, and to find what happens to the system, we must solve the time-dependent Schrödinger equation using this H . The approximate method of solution that we will describe is called **time-dependent perturbation theory**. To simplify our discussion, we will suppose that the perturbation W is switched on for a short time Δt , in which case, we can use the approximate solution (11.17) to find $\Psi(\Delta t)$ after the perturbation is switched off:

$$\begin{aligned} \Psi(\Delta t) &\approx \Psi(0) + \frac{\Delta t}{i\hbar} H \Psi(0) = \psi_n + \frac{\Delta t}{i\hbar} (H_0 + W)\psi_n \\ &= \left(1 - \frac{iE_n \Delta t}{\hbar}\right) \psi_n - \frac{i \Delta t}{\hbar} W \psi_n \quad (11.27) \end{aligned}$$

Comparing with (11.18), you can see that the first term here is just what would have evolved in the absence of any perturbation; in particular, it is proportional to ψ_n , so it has not changed the physical state of the system at all. On the other hand, the second term is an additional term, which can represent a small component of a *different* stationary state ψ_m . That is, because of this extra term resulting from the perturbation W that we switched on, there is a nonzero probability that we will find the system in a different state ψ_m . Before we discuss this important result any more, let us illustrate it with a simple example.

Example 11.3

An electron is initially in the ground state ψ_1 of a one-dimensional rigid box. At time $t = 0$ we switch on an electric field \mathcal{E} in the x direction, giving the electron an additional potential energy $W = e\mathcal{E}x$. Describe the form of the electron's wave function a short time Δt later when we switch the perturbation off.

According to (11.27), with W replaced by $e\mathcal{E}x$, the wave function at time t is

$$\Psi(\Delta t) \approx A\psi_1 - \frac{ie\mathcal{E}\Delta t}{\hbar}x\psi_1 \quad (11.28)$$

where A denotes the multiplier in parentheses of (11.27). The first term is just a multiple of the original wave function ψ_1 and is what would have evolved in the absence of any perturbation. The second is the extra term, caused by the perturbing field \mathcal{E} . Its significance is illustrated in Fig. 11.2. Part (a) of that figure shows the original wave function ψ_1 . (It is convenient in this discussion to place the origin at the center of the well, so the edges are now at $\pm a/2$.) Part (b) shows the extra term of (11.28), which is proportional to $x\psi_1$. The amplitude of this function depends on the parameters of the problem, but is certainly small (since our approximate solution requires that Δt be small). The important point is that the extra term shown in Fig. 11.2(b) has almost exactly the shape of the first excited state ψ_2 , which is plotted for comparison in part (c). The main message of Figure 11.2 is that switching on the perturbation $W = e\mathcal{E}x$ has added on to the initial wave function ψ_1 a small term that closely approximates ψ_2 (times some constant), so that the full wave function now has the form

$$\Psi(\Delta t) \approx A\psi_1 + B\psi_2 \quad (11.29)$$

There is now a small probability — namely, $|B|^2$ — that a measurement of the system will find it in the state* ψ_2 .

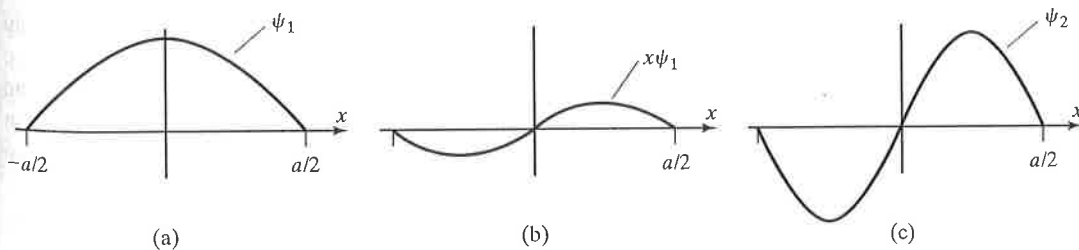


FIGURE 11.2

(a) The ground-state wave function ψ_1 for an electron in an infinite square well of width a , centered on the origin. (b) An electric field \mathcal{E} switched on briefly adds to the wave function of part (a) a small term proportional to $x\psi_1$. (c) The extra term shown in part (b) has almost exactly the shape of the excited-state wave function ψ_2 .

*As we will see in the next example, there is also a small probability of finding the electron in certain higher levels, $m > 2$, but these probabilities are so much smaller than that for $m = 2$ that the form (11.29) is an excellent approximation.

In general, the effect of switching on a perturbation W is to add to an initial wave function ψ_n the extra term of (11.27) proportional to $W\psi_n$, which may contain small components of several different wave functions ψ_m . For each $m \neq n$, there is therefore a definite probability $P(n \rightarrow m)$ that a system that started out in the state ψ_n will be found, after the perturbation has acted, in the state ψ_m . To evaluate this probability, we can expand the extra term as in (11.20)

$$-\frac{i \Delta t}{\hbar} W\psi_n = \sum_m A_m \psi_m \quad (11.30)$$

and according to (11.24), the coefficient A_m is given by the integral

$$A_m = -\frac{i \Delta t}{\hbar} \int_{-\infty}^{\infty} \psi_m^* W\psi_n dx \quad (11.31)$$

The **transition probability** $P(n \rightarrow m)$ is then just $|A_m|^2$, so

$$P(n \rightarrow m) = \left(\frac{\Delta t}{\hbar} \right)^2 \left| \int_{-\infty}^{\infty} \psi_m^* W\psi_n dx \right|^2 \quad (11.32)$$

The integral inside the absolute value signs is called the **matrix element** for the transition ($n \rightarrow m$). By evaluating this integral, we can find the probability for any transition in which we may be interested.

Example 11.4

Consider again the electron of Example 11.3, which is initially in the ground state of the infinite square well and is exposed to an electric field \mathcal{E} for a short time Δt . Find the probabilities $P(1 \rightarrow m)$ that it will subsequently be found in the level m for $m = 2, 3$, and 4.

The required probabilities are given by (11.32), with $W = e\mathcal{E}x$. Thus,

$$P(n \rightarrow m) = \left(\frac{e\mathcal{E} \Delta t}{\hbar} \right)^2 \left| \int_{-\infty}^{\infty} \psi_m^* x \psi_n dx \right|^2 \quad (11.33)$$

The evaluation of these integrals is a straightforward (though surprisingly tedious) exercise. The wave functions ψ_n are given in (11.21), although we have to rewrite them since we have moved our origin to the center of the well. You should have no difficulty in showing that, with the well centered on the origin, the wave functions with n odd become

$$\psi_n = \sqrt{\frac{2}{a}} \cos \frac{n\pi x}{a} \quad (n \text{ odd}) \quad (11.34)$$

whereas, for n even, they become

$$\psi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \quad (n \text{ even}) \quad (11.35)$$

Thus, the integral of (11.33) for the case that $n = 1$ and $m = 2$, is

$$\int_{-\infty}^{\infty} \psi_2^* x \psi_1 dx = \frac{2}{a} \int_{-a/2}^{a/2} \sin \frac{2\pi x}{a} x \cos \frac{\pi x}{a} dx \quad (11.36)$$

which can be evaluated (Problem 11.21) to give $16a/9\pi^2$. Thus,

$$P(1 \rightarrow 2) = \left(\frac{e\mathcal{E}\Delta t}{\hbar} \frac{16a}{9\pi^2} \right)^2 \quad (11.37)$$

In exactly the same way, we can evaluate

$$P(1 \rightarrow 4) = \left(\frac{e\mathcal{E}\Delta t}{\hbar} \frac{32a}{225\pi^2} \right)^2$$

some 150 times smaller than $P(1 \rightarrow 2)$. The transition $1 \rightarrow 3$ is more interesting: The integral of (11.33) is

$$\int_{-\infty}^{\infty} \psi_3^* x \psi_1 dx = \frac{2}{a} \int_{-a/2}^{a/2} \cos \frac{3\pi x}{a} x \cos \frac{\pi x}{a} dx \quad (11.38)$$

There is no need to evaluate this integral: It is easy to see that the integrand is an odd function of x ; that is, its value at $-x$ is the negative of its value at x . Since the range of integration is the symmetric interval $-a/2$ to $a/2$, the contribution from any point x is cancelled by that from $-x$, and the integral is exactly zero. Thus, at least within our approximations, the probability for the transition $1 \rightarrow 3$ is zero, $P(1 \rightarrow 3) = 0$.

It is easy to see that this last result applies to any transition $n \rightarrow m$ with both n and m odd, and likewise with both n and m even. Because of the symmetries of the sine and cosine functions, the integrals concerned are zero, and

$$P(\text{odd} \rightarrow \text{odd}) = P(\text{even} \rightarrow \text{even}) = 0 \quad (11.39)$$

Selection Rules

The final result of the last example is an instance of an important phenomenon. It often happens that because of the symmetries of the wave functions, the integral involved in the transition probability $P(n \rightarrow m)$ is exactly zero. When this happens, we say that the transition concerned is a **forbidden transition**, and within the approximations used the probability $P(n \rightarrow m)$ is zero. Strictly speaking, this does not mean that the transition is impossible, since the expression (11.32) is only an approximation. Nevertheless, it is usually the case that the probability of forbidden transitions is small compared to other “non-forbidden” transitions. With this terminology, we can say that in the last example, transitions between any two odd levels (and likewise between any two even levels) are forbidden. A rule of this kind, specifying a whole family of forbidden transitions, is called a **selection rule**. Selection rules are very useful in deciding which transitions of a quantum system should be observable with appreciable probability.

Upward and Downward Transitions

In our example of the electron in the infinite square well, we assumed that the system started out in its ground state, so that all other states had higher energy, and the only possible transitions were upward transitions to a higher level. However, the result (11.32) applies equally whether the level n is lower than the level m or vice versa. Thus, if the electron were initially in the level $n = 2$,

switching on an electric field could cause it to drop downward to the ground state. Transitions from a higher to a lower level usually result in the emission of a photon. Thus, a downward transition caused by an externally applied field is called stimulated emission — a process that plays a key role in the operation of lasers, as we discuss shortly.

Perturbations that Last a Long Time

So far we have considered only a perturbation that is switched on for a short time Δt . It often happens that the external influence that causes atomic transitions lasts for a long time. For example, we could place the atom in a steady beam of light, in which case the atom would feel oscillating electric and magnetic fields that last for hours or even days. To handle this, we need to extend our time-dependent perturbation theory to cover longer time intervals. This extension is unfortunately beyond our scope here, but the main results can be easily described. It turns out that the transition probability is given by a formula very similar to (11.32) except that (as you might guess) it involves an integral over time. In particular, it remains true as in (11.32) that the transition probability is proportional to the absolute value squared of a “matrix element”

$$P(n \rightarrow m) \propto \left| \int_{-\infty}^{\infty} \psi_m^* W \psi_n dx \right|^2 \quad (11.40)$$

Thus, many of the ideas discussed above still apply. For example, when symmetries of the perturbation W and the wave functions imply that the integral in (11.40) is zero, we say that we have a selection rule, and the transition concerned is said to be forbidden.

An important example of a perturbation that lasts a long time is the oscillating electric field associated with electromagnetic radiation that we could direct at an atom. In this case the perturbation would have the form $e\mathcal{E}_0 x \cos \Delta t$, or, a little more generally,

$$W(\mathbf{r}) \cos \omega t \quad (11.41)$$

where ω is the angular frequency of the applied radiation. In this case the expression inside the absolute value signs of (11.40) must include a time integral as well. This integral over time turns out to be negligibly small unless the frequency ω of the applied radiation is close to the value

$$\omega = |E_m - E_n|/\hbar \quad (11.42)$$

That is, transitions occur appreciably only if $\hbar\omega$ is close to the energy difference between the two levels concerned. This result agrees with our knowledge that a photon carrying energy $\hbar\omega$ can cause an upward transition ($n \rightarrow m$) with $E_m > E_n$ only if $\hbar\omega$ is equal to the difference $E_m - E_n$, in which case the photon is absorbed. However, there is a second important possibility: If $E_m < E_n$, the downward transition $n \rightarrow m$ can occur, provided $\hbar\omega$ is close to $E_n - E_m$. In this case, energy conservation requires that the atom release energy, which it does by emitting a photon of energy $\hbar\omega = E_n - E_m$, so that the incident radiation *gains* one photon. Since the emission of this photon is caused, or *stimulated*, by the applied radiation, it is called **stimulated emission of radiation**. Both of these possibilities are illustrated schematically in Fig. 11.3. The possibility of stimulated emission, depicted in part (b), is the basis of the laser, as we discuss in Sections 11.9 and 11.10.

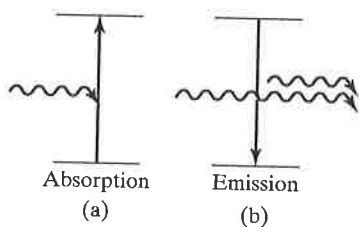


FIGURE 11.3

Schematic diagram of absorption and stimulated emission. (a) An incident photon is absorbed by the atom, which makes an upward transition. (b) In stimulated emission one photon striking the excited atom stimulates the emission of a second photon, causing the atom to make a downward transition.

11.6 A Brief Review

Let us review briefly the results of the previous section. If you read that section, this review may help to reinforce its ideas, and if you didn't, this will establish the ideas you need to take on faith. The central result is this: Suppose that an isolated quantum system is initially in one of its stationary states ψ_n . Suppose next that the system is subjected to some outside perturbation giving it an additional potential energy, which could be a static $W(\mathbf{r})$ or an oscillatory $W(\mathbf{r}) \cos \omega t$. Then, at any later time, there is a definite probability $P(n \rightarrow m)$ that the system, which was in the state ψ_n , will be found in a different state ψ_m . This transition probability can be calculated by the method of time-dependent perturbation theory and is found to have the form (11.40)

$$P(n \rightarrow m) \propto \left| \int \psi_m^* W \psi_n dV \right|^2 \quad (11.43)$$

written here in the form appropriate for an electron in three dimensions, with $dV = dx dy dz$. If the perturbation is the oscillating electric field of electromagnetic radiation, the transition ($n \rightarrow m$) can be an upward transition, resulting in the absorption of one of the incident photons, or a downward transition, resulting in the **stimulated emission** of another photon of the same frequency. In either case, transitions associated with electromagnetic radiation and involving the emission or absorption of photons are called **radiative transitions**.

It often happens that the integral (or "matrix element") in (11.43) can be shown to be zero on the grounds of symmetries of the perturbation W and the two wave functions. In this case, we say that the transition ($n \rightarrow m$) is **forbidden**, although this usually means only that the transition is much less probable than other "non-forbidden" transitions.

11.7 Spontaneous Emission

We have seen that radiation directed at an excited atom can cause it to make a downward transition, emitting a photon as it does so. However, we also know that an excited atom can make a downward transition even when it is isolated from any applied radiation. This process is called **spontaneous emission**. As we discussed in Section 11.3, an isolated atom in a truly stationary state, excited or not, would remain forever in that state. How, then, do excited atoms give off light without the aid of stimulating radiation? The answer is that the radiation required to stimulate downward transitions is *always* present, even when an atom is perfectly isolated from all apparent sources. The origin of this ubiquitous radiation has to do with the quantization of the electromagnetic field.

We know that the classical description of the electromagnetic field fails to explain the quantization of radiation. The complete quantum treatment of electromagnetic radiation is called **quantum electrodynamics** (or **QED**) and is beyond the scope of this book. However, we can easily describe its predictions relevant to our present discussion: At any given frequency, ω , the energy of radiation is quantized and can change only by integer multiples of $\hbar\omega$, corresponding to the now-familiar concept of photons. Furthermore, QED predicts that the lowest possible energy of the radiation field is *not* zero. That is, when one treats the electromagnetic field as a quantum system, its lowest energy level is found to be greater than zero.

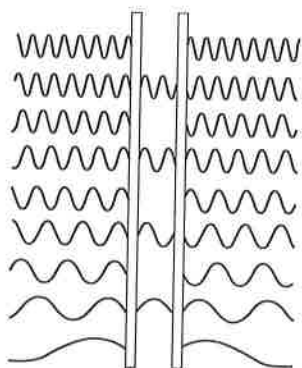


FIGURE 11.4

In the Casimir effect, zero-point radiation exerts forces on two uncharged parallel metal plates. Outside the plates, all wavelengths of radiation are allowed; between the plates, only certain discrete wavelengths are allowed (just as only certain discrete wavelengths are allowed on a stretched string). If the plates are close enough together, this difference produces a measurable inward force, due to the unequal radiation on opposite sides of either plate.

We have seen similar behavior in other quantum systems. For instance, the minimum energy of a particle in a box is nonzero. As we saw in Section 6.8, the reason for this is the Heisenberg uncertainty relation between position and momentum. A similar uncertainty relation applies to the electric and magnetic fields \mathcal{E} and B and implies that they cannot both be exactly zero, even in empty space. As a result, the electromagnetic field has a nonzero amplitude at every frequency. This minimum possible field is called the **zero-point**, or **vacuum-fluctuation, field**.

In many situations the effects of the zero-point fields are negligible. Nevertheless, the presence of these fields can be demonstrated in an experiment first proposed by the Dutch physicist Hendrik Casimir. As illustrated in Fig. 11.4, this experiment measures a tiny force pushing two uncharged parallel metal plates together. This force would not exist if space were not filled by the zero-point fields.

For our present purposes, a far more important consequence of the zero-point fields is their effect on an isolated atom. If the isolated atom is in an excited state, the zero-point fields can stimulate it to emit a photon and drop to a lower state. Since this process occurs without externally applied radiation, it is called spontaneous emission. However, you should recognize that stimulated and spontaneous emission are essentially the same phenomenon. Stimulated emission is caused by an externally applied field. Spontaneous emission is just stimulated emission at its lowest possible rate, which occurs when the electromagnetic fields are at their minimum possible (zero-point) level.

Although the zero-point fields cause spontaneous downward transitions, conservation of energy does not allow them to cause upward transitions. For an upward transition to occur, the energy gained by the atom must be balanced by a loss of energy from the radiation field. This is clearly impossible when the energy of the radiation field is already at its lowest possible value. In the special case of an atom in its ground state, this means that there can be no spontaneous transitions at all. Upward transitions cannot occur because energy could not be conserved, and downward transitions do not occur because there are no lower states. Thus the ground state of an isolated atom is truly stable.

11.8 Atomic Selection Rules

Let us summarize our findings as they apply to an electron in an atom exposed to radiation (either radiation that is applied externally or just that of the zero-point fields). The most important effect of the radiation comes from the oscillatory electric field of the form $\mathcal{E} = \mathcal{E}_0 \cos \omega t$. (Of course, there is also an oscillatory magnetic field, but its effect is usually much less important.) Thus, the dominant perturbation has the form

$$e\mathcal{E}x = e\mathcal{E}_0x \cos \omega t \quad (11.44)$$

(This is if the electric field points in the x direction. In three dimensions, it can, of course point in any direction.) If the electron is initially in a state ψ_n , then the probability $P(n \rightarrow m)$ that at a later time it will be found in a state ψ_m is given by (11.43) as

$$P(n \rightarrow m) \propto \mathcal{E}_0^2 \left| \int \psi_m^* x \psi_n dV \right|^2 \quad (11.45)$$

This gives the probability of the transition ($n \rightarrow m$) if the perturbing electric field is in the x direction. If the field is in the y direction, we must simply replace x by y , and so on. And if the perturbing radiation is unpolarized (as is certainly the case for the zero-point radiation on an otherwise isolated atom), we must average over all three directions.

As we mentioned earlier, if the matrix element in (11.45) is zero, we say that the transition ($n \rightarrow m$) is forbidden, meaning that at least within the approximations that led to (11.45) the probability $P(n \rightarrow m)$ is zero. It is sometimes surprisingly easy to show that this is the case. For example, suppose that the initial state ψ_n is an s state (with $l = 0$) and suppose that we are interested in the possibility of a transition to another s state ψ_m . In this case both of the wave functions in (11.45) are spherically symmetric. Therefore, the contribution to the integral from any point (x, y, z) is exactly cancelled by the contribution from the point $(-x, y, z)$, and the whole integral is automatically zero. That is, all transitions from one s state to another are forbidden (meaning in practice that such transitions are anyway extremely improbable). This is a particular case of a more general **selection rule** that any transition ($n \rightarrow m$) is forbidden, unless $l_m - l_n = \pm 1$, a rule that we usually state as:

$$\text{any transition } (n \rightarrow m) \text{ is forbidden unless } \Delta l = \pm 1 \quad (11.46)$$

The proof of this selection rule is beyond our scope here, but is illustrated by some simple cases in Problems 11.26, 11.30, and 11.31. The rule is easy to understand if we recall that the photon has spin 1. (That is, it has an intrinsic angular momentum of magnitude $\sqrt{s(s+1)}\hbar$ with $s = 1$.) This means that when a photon is emitted by or absorbed into an atom, it must change the atom's angular momentum by 1 unit, as in (11.46). (This is actually a quite subtle argument — see Problem 11.31.)

The selection rule (11.46) is illustrated in Fig. 11.5, which shows some of the allowed (that is, non-forbidden) transitions in the hydrogen atom. While there are many such allowed transitions, there are even more that are forbidden. For example, all transitions of the form $s \rightarrow s$ are forbidden (as we proved above) and so are any of the form $s \leftrightarrow d$ or $s \leftrightarrow f$. Similarly, both $p \rightarrow p$ and $p \leftrightarrow f$ are forbidden. We can express all this compactly by saying that a state in one column of the energy-level diagram can make transitions only to states in one of the adjacent columns. In particular, s states can make transitions only to p states. Other examples of selection rules appear in Problems 11.25, 11.27,

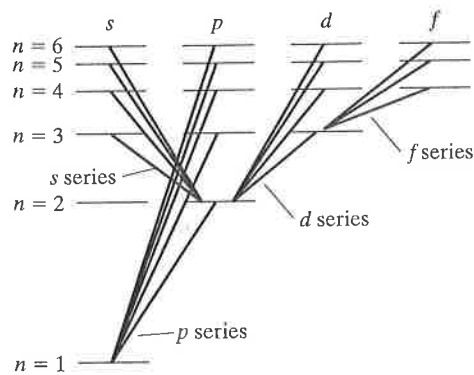


FIGURE 11.5

Some of the allowed transitions observed in the hydrogen atom. Note that each involves a change of l by one unit, as is found to be the case for *all* allowed transitions. Note also that the traditional labels s (sharp), p (principal), d (diffuse), and f (fundamental) were originally applied to transitions, not levels.

and 11.30. For reference, we have collected together some of the more important selection rules in Table 11.1.

TABLE 11.1

Some of the selection rules that apply to transitions of electrons in an atom. Each rule is stated in the form of a condition that must be met if a transition is to be allowed (that is, occur with significant probability). For example, the first rule, $\Delta l = \pm 1$, means that only transitions for which $\Delta l = l_f - l_i = \pm 1$ are allowed. The quantum number s_{tot} identifies the magnitude of the total spin of all the electrons; similarly j_{tot} gives the magnitude of the total angular momentum $\sum(\mathbf{L} + \mathbf{S})$.

Quantum Number	Selection Rule	Reference
l (magnitude of \mathbf{L})	$\Delta l = \pm 1$	Eq. (11.46)
m (z component of \mathbf{L})	$\Delta m = 0$ or ± 1	Problem 11.30
s_{tot} (total spin $\sum \mathbf{S}$)	$\Delta s_{\text{tot}} = 0$	Problem 11.25
j_{tot} [total spin + orbital $\sum(\mathbf{L} + \mathbf{S})$]	$\Delta j_{\text{tot}} = 0$ or ± 1	Problem 11.27

Metastable States

Looking back at Fig. 11.5, you can see that the $2s$ level of hydrogen has no allowed downward transitions because there is no $l = 1$ level below the $2s$ level. This would seem to imply that the $2s$ state is perfectly stable. In fact, there exists other processes that de-excite the $2s$ state, such as a collision with other atoms.* However, these processes all occur very slowly, and the lifetime of the $2s$ state is exceptionally long. There are many other atoms that have similar excited states with no allowed downward transitions. When this occurs the long-lived excited state is called a **metastable state**. These states are important in the operation of many lasers, as we see in the next section.

11.9 Lasers

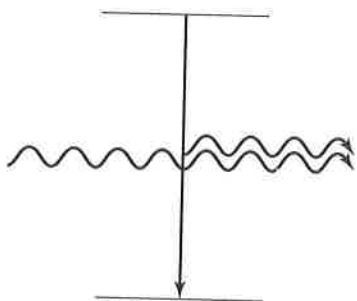
Lasers are made possible by the abstruse-seeming phenomena of stimulated emission and metastable states. Today lasers play such a large role in our lives it is hard to imagine life without them. In the last two sections of this chapter we will describe how several kinds of lasers work and discuss briefly some of their uses.

The process of stimulated emission is illustrated in Fig. 11.6. A single photon of the correct frequency can stimulate an excited atom to drop to a lower level and emit another photon of the same frequency. If there are many excited atoms present, each of these two photons can go on to stimulate two more emissions, producing four photons in all. As long as the majority of atoms are still in the excited state, this process can continue in a cascade, giving

$$1 \text{ photon} \rightarrow 2 \text{ photons} \rightarrow 4 \text{ photons} \rightarrow 8 \text{ photons} \rightarrow \dots$$

and so on. In other words, the process of stimulated emission can produce a dramatic *amplification* of a beam of photons. The **laser** is a device that exploits

* Another such process, which occurs even in isolated atoms, is the simultaneous emission of two or more photons, but this occurs extremely rarely.

**FIGURE 11.6**

Incident radiation stimulates a transition from an excited state to a lower state. The emitted radiation is exactly in phase with the incident radiation.

this possibility to amplify light of a definite frequency. The name laser is an acronym for "Light Amplification by the Stimulated Emission of Radiation."

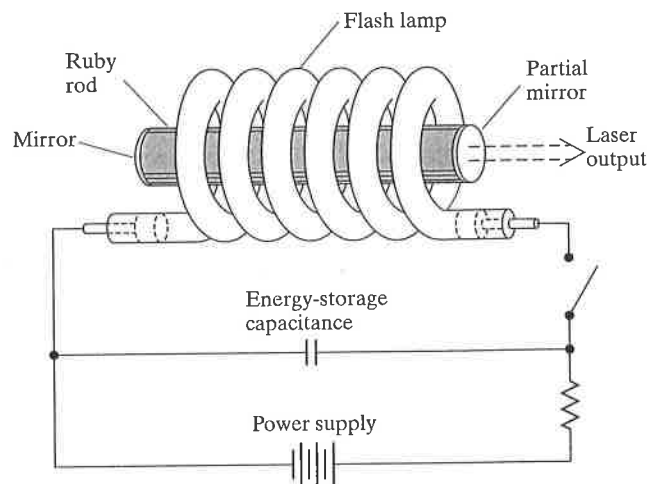
In stimulated emission the time dependence of the oscillation induced in each atom is in lockstep with the stimulating radiation. This means that the emitted radiation is exactly in phase with the radiation causing the transition, as suggested in Fig. 11.6. This **coherence** of the emitted photons means that the resulting light wave is an almost perfect sinusoidal wave. This contrasts with the **incoherent** light from ordinary sources where the phase of each atom's radiation is random with respect to all the others. It can also be shown that this coherence means that the laser beam can have a very well defined direction in space.

We have described the amplification of light in a laser as caused by stimulated emission from excited atoms. Normally, however, most atoms are in their ground state, and radiation that strikes atoms in their ground state will be absorbed, not amplified. To achieve amplification, we must arrange that a majority of the atoms are in an excited state. This reversal of the normal population of the levels is called a **population inversion**. Different lasers use different means to achieve population inversion, as we describe in the following accounts of some important types of lasers. Some lasers produce short pulses of light, while others produce a continuous beam. Since the details of operation of these two types are somewhat different, we describe them in turn, starting with the pulsed laser.

Pulsed Lasers

The first successful laser for visible light* was a pulsed laser, developed by the American physicist Theodore Maiman in 1960. The essential element of this laser was a ruby rod containing chromium ions, which have a metastable state 1.79 eV above the ground state. Transitions between these two states produce photons with a wavelength $\lambda = 694$ nm, in the deep red portion of the visible spectrum.

The general design of the ruby laser is shown in Fig. 11.7. In the first step of operation, a brilliant burst of light from a flash lamp causes most of the Cr



Charles Townes

(born 1915, American)



The principle of the laser was first developed for microwaves, the corresponding device being the maser (an acronym for Microwave Amplification by Stimulated Emission of Radiation). Townes had a large part — both theoretical and practical — in development of the maser. He also contributed substantially to the laser (which amplifies light). Both masers and lasers have revolutionized almost every branch of science; for example, the maser is the basis of the atomic clock. For his role in all this, Townes shared the 1964 Nobel Prize in physics. In the photo, Townes is standing next to a maser amplifier for radio astronomy.

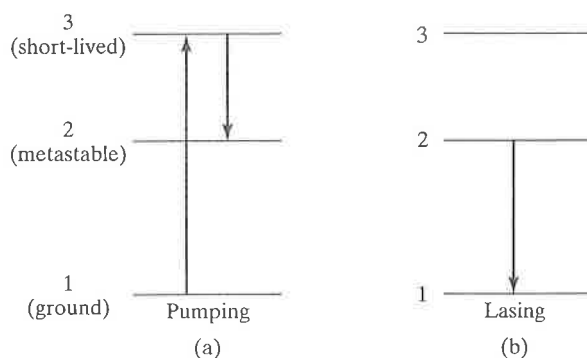
FIGURE 11.7

Schematic drawing of a pulsed ruby laser. The flash lamp excites Cr ions in the ruby rod. The mirrors cause light to reflect back and forth in the rod to increase the probability of stimulating further emission. One mirror is not 100% reflective, so some light escapes to form the external beam.

*We should mention that the first use of stimulated emission to amplify any kind of radiation was with microwaves. Devices that amplify microwaves in this way are called masers.

FIGURE 11.8

The relevant levels of the chromium ions in a ruby laser are: (1) the ground state, (2) the metastable state at 1.79 eV, and (3) a short-lived state. (a) Absorption of flash-lamp light elevates many Cr ions to the short-lived state, which promptly decays to the metastable state, where a large population builds up. (b) The process of stimulated emission then causes the laser transition to occur in many Cr ions.



Theodore Maiman

(born 1927, American)



After getting his BA from the University of Colorado and his PhD from Stanford, Maiman joined the Hughes Research Laboratory in Miami, where he developed the first working laser in 1960. He subsequently founded several independent companies, one of which, Korad Corporation, became the leading developer and manufacturer of high-power lasers.

ions in the ruby rod to be excited out of their ground state. The majority of these excitations populate a short-lived state that makes a prompt transition to the metastable state at 1.79 eV, as shown in Fig. 11.8. (The process of moving the ions from their ground state to the metastable level is called **pumping**.) Because the metastable level is relatively long-lived (about 4 ms) a majority of the Cr ions are caught, briefly, in this level. Once the number of ions in the metastable state is greater than the number in the ground state ($N_2 > N_1$), any light produced by a spontaneous $2 \rightarrow 1$ transition is amplified by stimulated emission from the excited ions it encounters. As this amplified light sweeps through the rod, it rapidly de-excites many excited ions and forms an intense, short pulse of light. This cascade can occur only if there is a population inversion ($N_2 > N_1$), for if $N_2 \leq N_1$, then absorption exceeds or equals emission, killing the amplification process. When the majority of the ions have returned to their ground state, the entire process can be repeated.

To give the light ample opportunity to stimulate emission, it is reflected back and forth along the rod, whose ends are polished and silvered to form two mirrors.* One of these mirrors is made partially transparent to allow light to escape and form the external beam of laser light. In the original laser the xenon flash lamp was powered by a capacitor bank of several hundred microfarads charged to several kilovolts. The principal parts of this laser can be seen in the photograph in Fig. 11.9.

There are three major differences between the pumping light from the flash lamp and the light produced by the laser action itself. First, the flash lamp emits a broad range of wavelengths (white light), while the laser light is concentrated in a narrow spectral line at $\lambda = 694$ nm. Second, the flash lamp light is incoherent, while the laser light is coherent. Third, the flash lamp light radiates in all directions, whereas the laser beam can be designed to have a narrowly defined direction in space — an important property in many applications.

Since the development of the ruby laser, a variety of other materials have been found that can be pumped to a metastable level and used to make pulsed lasers. These lasers vary widely as to total energy and pulse length. For the ruby laser, a representative value for the energy of a single pulse is on the order of 10 J. Since each pulse has a typical duration of order[†] 100 μ s, this gives an instantaneous power of order 100 kW. This high instantaneous power

*The distance L between the mirrors must satisfy the condition $L = n\lambda/2$, to ensure constructive interference of the multiple reflected waves.

† As we discuss in Section 11.10, this pulse is really a cluster of much shorter pulses.

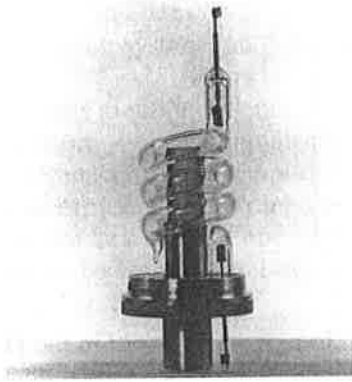


FIGURE 11.9

The original laser, built by Maiman at the Hughes Research Lab. The ruby rod, about 1 cm in diameter, can be seen inside the coiled flash lamp.

is the basis of many laser applications: Small pulsed lasers are used in medicine to cause coagulation and to suture tissues by forming tiny scars. Larger lasers are widely used in industry for welding, perforating, and machining. Yet larger versions are being investigated for military use.

Continuous Lasers

For many purposes, it is convenient to have a continuous wave of laser light. Most **continuous-wave** (or **CW**) lasers use *four* levels, instead of the three levels used in the ruby laser. This avoids an important problem with any three-level laser: In a three-level laser, more than half the atoms must be excited out of their ground state to achieve the necessary population inversion, with $N_2 > N_1$. Since nearly all atoms are normally in the ground state, a great deal of energy must be supplied by the pumping flash to accomplish this. A **four-level laser** greatly reduces this requirement by making the lower level of the laser transition an excited state which is normally empty. As shown in Fig. 11.10, the laser transition from level 3 to level 2 avoids the heavily populated ground state so that the condition of population inversion ($N_3 > N_2$ in this case) is more easily achieved. Because of their lower power requirements, such lasers can operate continuously.

A popular type of continuous four-level laser is the helium–neon laser. This uses a mixture of helium and neon gases and produces red light with $\lambda = 633$ nm. The uppermost level of this laser is in the helium atoms, and the other three are in the neon, as shown in Fig. 11.11. The He atoms are pumped by high-speed electrons rather than by a flash lamp. This is achieved by a

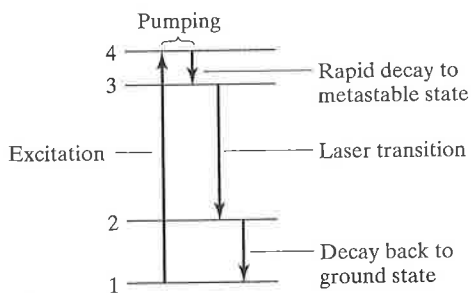


FIGURE 11.10

In a four-level laser the laser transition is between levels 3 and 2. Since few atoms are in level 2, the condition $N_3 > N_2$ is readily achieved.

FIGURE 11.11

The level initially pumped in the He–Ne laser is in the He atoms. Collisions transfer this energy to a level in the Ne atoms, which then produce stimulated emission, terminating in a nearly empty excited state.

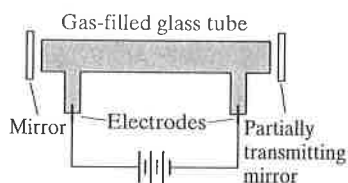
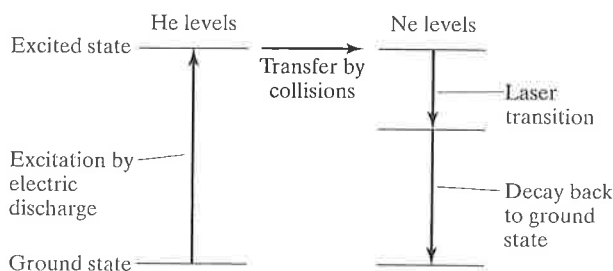


FIGURE 11.12

Key components of an He–Ne laser are the gas-filled glass tube, mirrors, and power supply.

steady electric discharge (like that in a neon sign) within the He–Ne mixture. The energetic electrons in the discharge strongly excite a metastable level in He whose energy is very close to one of the excited levels of Ne. In collisions between the excited He atoms and unexcited Ne atoms, there is a high probability for transfer of the excitation energy, with the He atom dropping to its ground state while the Ne atom is excited. (The upper level in Ne is not so readily excited directly by fast electrons, hence the need for He and the transfer process.) The laser transition occurs in the Ne, producing the characteristic red light at 633 nm.

Not many atoms are required to be in the uppermost Ne level because the level below it rapidly empties by a fast transition to the ground state. The condition $N_3 > N_2$ is therefore achieved with only modest power requirements for the electric discharge.

Figure 11.12 is a schematic of an He–Ne laser. The gas mixture fills a glass tube fitted with electrodes to produce a continuous electric discharge. As in the ruby laser, mirrors reflect light back and forth to increase the opportunity for stimulated emission. The external beam passes out through one of the mirrors, which is partially transmitting. Typical small models produce a light beam with a power of 10^{-3} W and consume a few watts of electric power.

Applications

The advent of the laser has totally transformed research in many branches of science. The sharply defined frequency of laser light has made possible measurements of quantized energy levels in atoms, molecules, and solids, with unprecedented accuracy. Because laser light is so intense, measurements can be made with very small samples — transitions in *single* atoms have even been observed. Using lasers with pulses that last just a few femtoseconds ($1 \text{ fs} = 10^{-15} \text{ s}$), chemists can follow the detailed evolution of chemical reactions. The momentum of the photons in a laser beam can be used to cool a gas to temperatures of order 10^{-6} kelvin (Chapter 13), but also to confine the plasma in a fusion device at higher than 10^6 kelvin (Chapter 17).

Lasers also allow extremely accurate measurements of distances. For example, in the lunar ranging experiment, the distance from the earth to the moon is measured by timing the round trip of a laser pulse that is fired at the moon and reflected back by a mirror placed there by Apollo astronauts; in this way changes in the earth–moon distance of a few centimeters can be detected. Similarly, geophysicists can monitor the extremely slow motions of the tectonic plates that comprise the earth's crust. Laser interferometers (described in Section 11.10) allow length measurements with accuracies of order one hundredth of the wavelength of light (that is, accuracies of a few nanometers). With the help of lasers, the speed of light can be measured so accurately that it has now become a defined constant, in terms of which the

SI meter is specified. And lasers play an essential role in the atomic clocks on which our timekeeping system (an essential part of the global positioning system) depends.

We have already mentioned some commercial applications of lasers, and it is easy to list more — in communications using optical fibers, in sound recording using compact discs, in computing, and so on. In medicine, lasers are used for cancer therapy and for welding tissue (in the human retina for example). To conclude, we mention just one more application, the laser gyroscope, which has replaced the conventional gyroscope in many commercial aircraft and is described in Fig. 11.13.

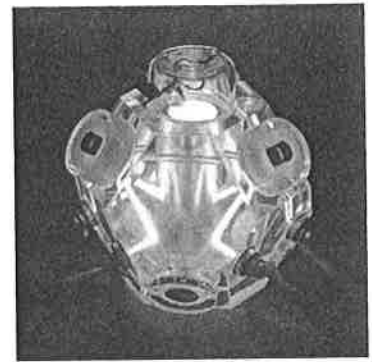


FIGURE 11.13

A laser gyroscope, consisting of three perpendicular, ring-shaped lasers, all bored in a single 5-inch cube of glass. Each ring contains two laser beams, rotating in opposite directions. The interference between the two beams is extremely sensitive to rotations of the ring, and with three such rings, the device can be used like an ordinary gyroscope to keep track of an aircraft's orientation.

11.10 Further Properties of Lasers*

*This section can be omitted without significant loss of continuity.

Angular Divergence

The directional properties of laser beams depend on the paths followed by light within the laser cavity (the space between the two mirrors). Figure 11.14(a) shows light reverberating back and forth in a straight line between the mirrors as its amplitude is increased by stimulated emission from excited atoms. A laser operating in this manner is called a single-mode laser. If the laser medium were perfectly homogeneous and the mirrors perfectly flat, the external beam from single-mode operation would be nondiverging if light were not a wave. However, because of the wave nature of light, diffraction occurs and causes the light to diverge slightly. The minimum angle of divergence, $\delta\theta$ (defined as the angle from center to first minimum), depends on the diameter d of the beam and is given by the diffraction relation*

$$\delta\theta \approx \frac{\lambda}{d} \quad (11.47)$$

where λ is the wavelength of light (and $\delta\theta$ is measured in radians). With typical laser dimensions, the actual angle of divergence $\delta\theta$ is of order 10^{-4} rad, which is extremely small compared with the spreading from most other light sources. Nevertheless, over large distances, even laser beams suffer appreciable spreading, which reduces their intensity, as we see in the following example.

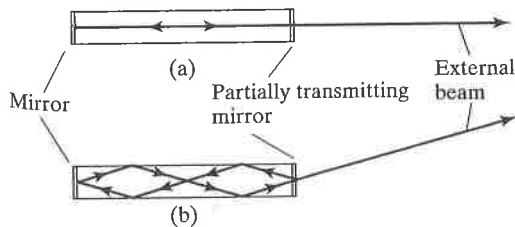
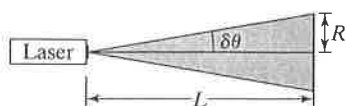


FIGURE 11.14

(a) A laser operating on only one mode, in which light reflects back and forth along a straight path between the mirrors. (b) A more complicated mode, which produces light traveling at an (exaggerated) angle.

*For a circular aperture, this formula is generally quoted as $\delta\theta = 1.22 \lambda/d$. For our purposes of estimation, we have dropped the factor of 1.22.

**FIGURE 11.15**

The radius of a laser beam at a distance L is approximately $R \approx L \delta\theta$. (Because R is much larger than the original beam size, we can neglect the latter.)

Example 11.5

A single-mode He-Ne laser ($\lambda = 633 \text{ nm}$) with an initial beam diameter of 3 mm has an angular divergence limited only by diffraction. Find the approximate diameter of the beam at a distance of 300 m. By what factor is the intensity reduced at this distance?

According to (11.47), $\delta\theta \approx \lambda/d$; therefore, as shown in Fig. 11.15, the radius R of the beam at a distance L is

$$R \approx L \delta\theta \approx \frac{L\lambda}{d}$$

With $L = 300 \text{ m}$, this gives

$$R = \frac{(300 \text{ m}) \times (6.3 \times 10^{-7} \text{ m})}{3 \times 10^{-3} \text{ m}} \approx 60 \text{ mm}$$

Therefore, the beam diameter is about 120 mm.

Since intensity is power per unit area, it is inversely proportional to the cross-sectional area of the beam. Thus the ratio of final to initial intensities is

$$\frac{I_f}{I_i} = \frac{\pi d_i^2/4}{\pi d_f^2/4} = \left(\frac{d_i}{d_f}\right)^2 \approx \frac{1}{1600}$$

This loss of intensity is sufficiently large that it must obviously be taken into account when planning to use lasers over large distances.

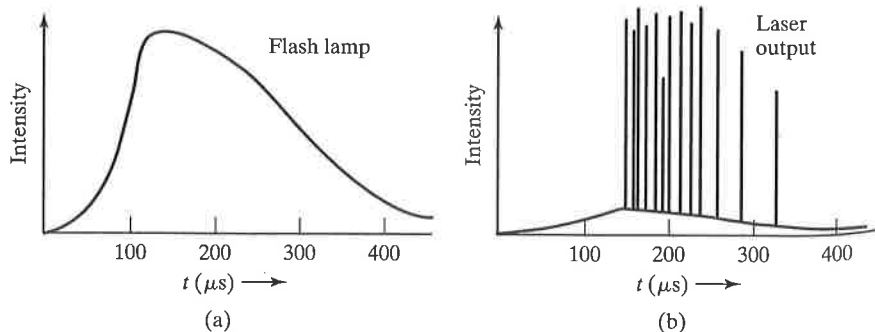
More complex modes, like that in Fig. 11.14(b), can also occur. These modes produce nonaxial light that increases the angular divergence of the laser beam; typically, $\delta\theta \approx 10^{-3} \text{ rad}$. A small aperture can be inserted in the laser cavity to limit laser action to the axial mode and hence reduce the angular divergence, but this usually results in a loss of total power.

Time Dependence of Pulsed Laser Beams

In a pulsed laser the flash lamp that initiates the pulse usually produces white light over a period of several hundred microseconds, as shown in Fig. 11.16(a). The output of the laser generally consists of a series of spikes, as in Fig. 11.16(b). The occurrence of these brief spikes is easily understood. Initially, in the first 100 μs or so, $N_2 < N_1$, so laser action cannot occur. Only spontaneous emission, which is incoherent, occurs during this time. When N_2

FIGURE 11.16

(a) The flash-lamp intensity in a pulsed laser typically lasts several hundred μs . (b) The output of the laser contains a weak background of incoherent spontaneous emission and a series of intense spikes of coherent light.



finally exceeds N_1 , laser amplification begins and the light level within the cavity builds rapidly. In fact, a "runaway" occurs because the rate of stimulated emission grows as the amplitude of the radiation increases. Very quickly, then, N_2 decreases until $N_2 < N_1$, and laser amplification ceases, typically in a fraction of a microsecond. However, the pumping radiation from the flash lamp is still present and again causes N_2 to increase until $N_2 > N_1$, producing another spike. This process continues until the level of pumping radiation falls below that which can make N_2 greater than N_1 .

Q-Switching

The size of the pulses produced by a ruby laser is limited because laser amplification begins as soon as $N_2 > N_1$ and ends quickly when N_2 drops back to $N_2 \approx N_1$. Larger pulses could be produced if N_2 could be greatly increased before laser amplification begins. This can be accomplished by preventing amplification temporarily until N_2 is large and then permitting laser action to occur. This method of producing single, very intense pulses is called **Q-switching**. The name originates in the term **quality factor**, generally abbreviated as Q . This parameter is a measure of the time light can reverberate within the laser cavity before it dies away.* If Q is large, the light loses energy (by scattering, for example) slowly and hence is easily able to stimulate laser amplification. If Q is small, the light loses energy quickly, and if Q is sufficiently small, laser action does not occur.

In the lasers described so far, the cavity is designed with high Q by using low-loss mirrors and a clear laser medium so that laser amplification is easily achieved. This causes laser action to begin very soon after $N_2 > N_1$ and limits the size of the pulse. However, if the Q of the cavity is reduced to a low level, laser action cannot occur even with all the atoms in the excited state. A very large pulse can then be produced by abruptly increasing Q (hence the name "Q-switching").

A variety of schemes are used for the temporary reduction of Q . The simplest method to understand uses a rotating mirror for one of the cavity mirrors as sketched in Fig. 11.17. The laser cannot operate until the mirrors are parallel. The flash lamp is fired a few hundred microseconds before the mirrors become parallel, so that the metastable level becomes highly populated. The pulse that occurs when the mirrors become parallel rapidly drains the population of level 2, producing a single extremely intense pulse. Figure 11.17(b) shows the pulse produced by a laser of this type with a peak power of 100 MW. (The peak power of such lasers can be as large as thousands of megawatts.)

Another Q-switching technique uses a dye contained in a transparent cell that is placed between the ruby rod and one of the mirrors. The color of a dye is due to its absorption of light at wavelengths corresponding to the energy difference between its ground state and a group, or *band*, of excited states. By use of a dye that absorbs strongly at the laser wavelength, Q is reduced so much that laser action cannot occur. However, after the flash lamp has operated for some time, virtually all of the dye molecules have been excited out of their ground state, so that absorption ceases.[†] At this point the dye becomes transparent, increasing Q , and a very large laser pulse occurs.

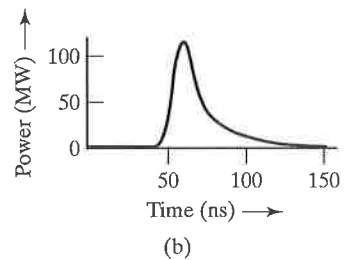
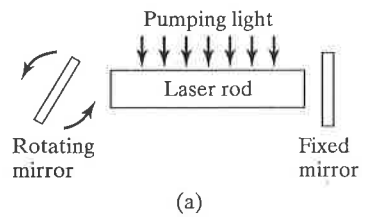


FIGURE 11.17

- (a) One type of Q-switching uses a rotating mirror, which prevents laser action until the instant the two mirrors are parallel.
- (b) Oscilloscope trace showing output power against time for a laser of this type.

*More precisely, Q is the ratio of the total energy in the cavity to the energy lost per cycle.

[†]A dye with this property is called a saturable dye and is said to have been bleached by absorption.

Coherence Length

To conclude this section, we return to the property of coherence of laser light as compared to light from other sources. In particular, we introduce the notion of **coherence length** as a measure of this property. To better understand the notion of coherence, let us consider first a perfectly sinusoidal wave. Such a wave has the property that its oscillations at any one point A are perfectly correlated, or coherent, with the oscillations at any other point B no matter how far A is from B . This means, for example, that if the distance $AB = n\lambda$ (with n an integer, small or large), the oscillations at A and B are exactly in step; and if we could somehow combine the waves from A and B , we would obtain perfectly constructive interference. Similarly, if $AB = (n + \frac{1}{2})\lambda$, the oscillations at A and B are exactly out of step and, if combined, would interfere perfectly destructively.

No real wave is perfectly sinusoidal. Among other things, it is found that light waves from most sources are produced with continual random shifts in their phase. (For instance, in an ordinary light filament, photons emitted in different regions are totally unrelated and have random relative phase.) This means that the relative phase of the oscillations at any two points A and B is continually and randomly changing. As one might expect, these random fluctuations are small if A and B are close together, but become steadily larger as the distance AB increases. Thus if $AB = n\lambda$ with n equal to a *small* integer, the oscillations at A and B remain very nearly in step and, if combined, would interfere almost completely constructively. On the other hand, if AB is sufficiently *large*, the relative phase of the oscillations at A and B fluctuates by 180° or more. In this case, even if AB is exactly $n\lambda$, the oscillations at A and B will sometimes be in step and sometimes completely out of step. If combined, these waves would interfere, sometimes constructively and sometimes destructively. Since the fluctuations are rapid, the observed intensity would be the average of the maximum and minimum intensities (namely, half the maximum). Evidently, once the distance AB is large enough, we would get this same result whether or not $AB = n\lambda$. Under these conditions we say that the oscillations at A and B are no longer coherent. Roughly speaking, we are going to define the coherence length of a wave as the largest distance AB for which the oscillations at A and B are still coherent.

We can test these ideas using a Michelson interferometer, as sketched in Fig. 11.18. The half-silvered mirror splits the beam into two parts, which travel out and back along arms of lengths l_1 and l_2 , and are then recombined and measured by a photocell. Since the two waves started as a single wave and traveled

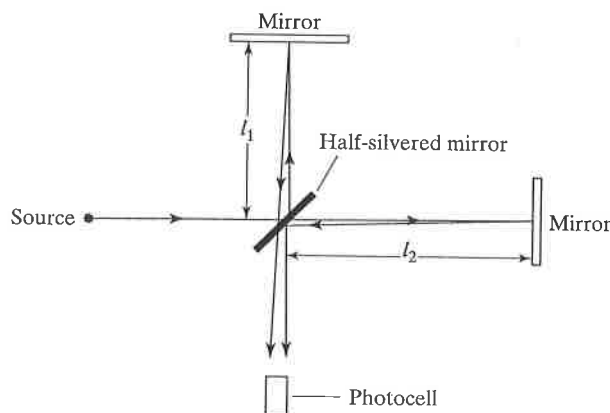


FIGURE 11.18

A Michelson interferometer. Light is split into two beams, which travel different distances, $2l_1$ and $2l_2$, before recombining.

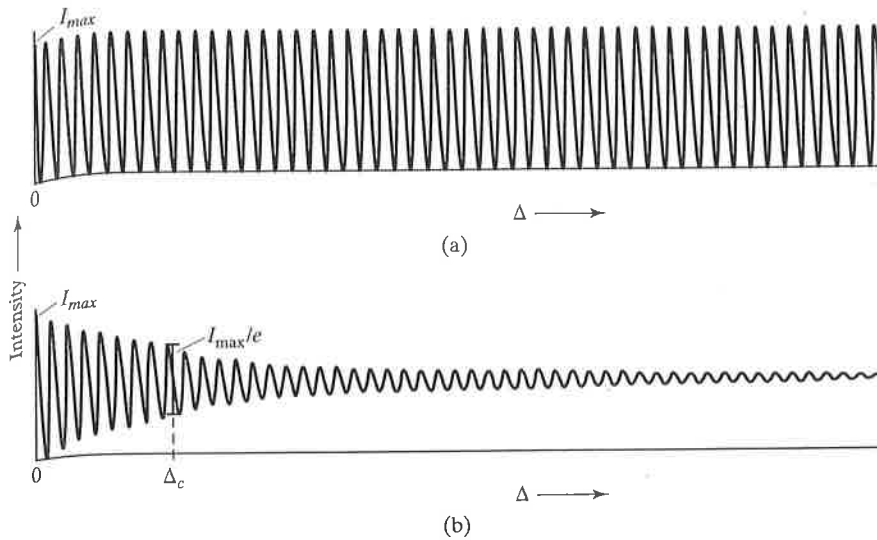


FIGURE 11.19

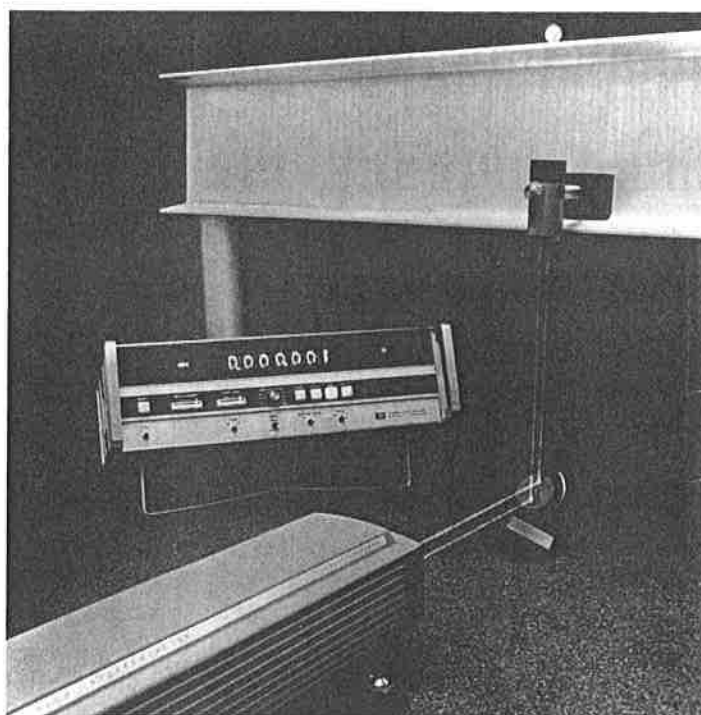
Intensity I as a function of path difference Δ in a Michelson interferometer. (a) If the light source were perfectly coherent, I would alternate between I_{\max} and 0 and back to I_{\max} each time Δ increased by one wavelength. (b) For any real source, one finds that as Δ increases, the coherence diminishes and the difference between the maxima and minima slowly decreases. The coherence length Δ_c can be defined as the value of Δ for which this difference has decreased to I_{\max}/e .

distances that differ by $\Delta = 2(l_1 - l_2)$, the photocell is in effect measuring the superposition of the oscillations at two points, A and B , in the same wave, the distance between A and B being the path difference Δ . By varying the length of either arm, we can study the interference as a function of the distance $AB = \Delta$.

If the original wave were perfectly sinusoidal, then as Δ increases, the observed intensity would alternate between maxima, all of the same height I_{\max} , and minima, $I_{\min} = 0$, as in Fig. 11.19(a). These alternations would continue no matter how large we made Δ . In reality, the wave is not perfectly sinusoidal and the situation is as shown in Fig. 11.19(b). When Δ is small, the intensity alternates between I_{\max} and 0, much as in Fig. 11.19(a). But as Δ increases, the two waves begin to lose their coherence, and the maxima begin to shrink and the minima start to grow. Finally, as $\Delta \rightarrow \infty$, the contrast between the maxima and minima disappears entirely, and I approaches the constant value $I_{\max}/2$.

It is clear from Fig. 11.19(b) that one cannot define a unique coherence length Δ_c , beyond which the two waves abruptly lose their coherence. Rather, the coherence fades out continuously, and one can, for example, define Δ_c as the distance at which the difference between successive maxima and minima has dropped to the value I_{\max}/e . This is the definition indicated in Fig. 11.19(b).

Before the advent of lasers, the coherence length of typical light sources was less than a millimeter or so, although lengths of order 10 cm could be achieved with difficulty. Since the coherence length of lasers can easily exceed 1 km, they have enormously increased the usefulness of interferometers for measuring distances. In a typical application one arm of the interferometer is kept fixed while the mirror on the other is moved through the distance that is to be measured. In this way, distances can be measured to a fraction of a wavelength. However, since interference cannot be observed once Δ is much more than Δ_c , the distances that can be measured cannot be much larger than Δ_c . Therefore, the laser has increased by many orders of magnitude the distances that can be measured using interferometers. Perhaps equally important, laser interferometers can measure displacements of objects that are far removed from the interferometer (and hence have $l_2 \gg l_1$), as with the arrangement shown in Fig. 11.20.

**FIGURE 11.20**

In this laser interferometer one of the paths is inside the device while the other extends to a mirror attached to the point whose displacement is to be measured. This photo shows a reading of 1 microinch, the displacement of the I-beam caused by the weight of a penny.

CHECKLIST FOR CHAPTER 11

CONCEPT	DETAILS
Radiation from a classical charge	$P = 2kq^2a^2/3c^3$ (11.1)
Inevitable collapse of classical atom	Lifetime $\sim 10^{-11}$ s (11.4)
Absence of radiation from stationary states	Static charge distribution (Sec. 11.3)
Transitions caused by perturbations	Upward = absorption of a photon Downward = emission of a photon
Hamiltonian operator	$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + U(x)$ (11.11) (with suitable generalizations)
Time-independent Schrödinger equation	$H\psi = E\psi$ (11.13)
Time-dependent Schrödinger equation	$i\hbar\partial\Psi/\partial t = H\Psi$ (11.16)
Completeness of the stationary-state wave functions	Any $\psi(x) = \sum A_n\psi_n(x)$ (11.20) $ A_n ^2$ = probability of finding system in state ψ_n $A_m = \int \psi_m^*\psi dx$ (11.24)
Orthogonality	$\int \psi_m^*\psi_n dx = 0$ for $m \neq n$ (11.23)
Normalization	$\int \psi_n^*\psi_n dx = 1$
Time-dependent perturbation theory	Theory of transitions due to a perturbation W (Sec. 11.5)
Transition probability	$P(n \rightarrow m) \propto \int \psi_m^*W\psi_n dx ^2$ (11.40) and (11.43)
Stimulated emission	Emission of a photon in a downward transition stimulated by applied radiation
Spontaneous emission	Emission of a photon in a downward transition of an isolated atom "stimulated" by the vacuum fluctuation fields

Selection rule
Metastable state
Lasers

Any general rule specifying forbidden transitions (Table 11.1)
An excited state with no allowed downward transitions (Sec. 11.8)
Pulsed and continuous (Sec. 11.9)
Angular divergence, time dependence, Q-switching, coherence length (Sec. 11.10)

PROBLEMS FOR CHAPTER 11

SECTION 11.2 (Radiation by Classical Charges)

- 11.1 • A charge q executes simple harmonic motion with position $x = x_0 \sin \omega t$. **(a)** Find P , the total power of the radiation emitted by this oscillating charge. **(b)** Show that the average power over one complete cycle is

$$\langle P \rangle = \frac{kq^2\omega^4x_0^2}{3c^3}$$

- 11.2 • In the antenna of a TV or radio station, charges oscillate at some frequency f and radiate electromagnetic waves of the same frequency. As a simple model of such an antenna, imagine that a single charge $q = 250 \text{ nC}$ is executing simple harmonic motion at 100 MHz with amplitude 0.3 m. (1 nC = 10^{-9} coulomb.) Use the result of Problem 11.1 to calculate the average total power radiated by this antenna.

- 11.3 • A certain cell phone transmitter radiates 3 W of power at about 900 MHz. **(a)** Find the rate of emission of photons by the transmitter. **(b)** In this case is there an appreciable difference between the correct quantum view and the classical picture in which the radiation is emitted continuously?

- 11.4 • Carry out the calculation of Example 11.1 using SI units throughout. [Note: If you use a calculator and exceed its maximum exponent range (usually $10^{\pm 99}$), you should calculate the mantissa and exponent separately.]

- 11.5 •• The formula (11.1) for the power P radiated by an accelerating charge q can be derived by the method of **dimensional analysis**. Since P would be expected to involve k, q, a , and c , we might reasonably guess that it should have the form

$$P = bk^l q^m a^n c^p \quad (11.48)$$

where b is some dimensionless number of order 1 (perhaps something like 4π) and where l, m, n , and p are unknown powers (and k is the Coulomb force constant $\approx 9 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2}$). By inserting their units into the five dimensional quantities in (11.48), you will get an equation that determines the unknown powers l, m, n , and p . Show that you obtain the correct form (11.1), except that the dimensionless number b cannot be determined.

- 11.6 •• The result (11.3) shows that a classical atom would collapse very rapidly. To get a very rough estimate of the time for a hydrogen atom to collapse completely, do the following: **(a)** You can find the rate at which the radius r shrinks,

$$\frac{dr}{dt} = \frac{dr}{dE} \frac{dE}{dt} \quad (11.49)$$

with dr/dE determined from (5.10) and $dE/dt = -P$. Find dr/dt when $r = a_B$. **(b)** Making the admittedly crude approximation that dr/dt remains constant, estimate roughly how long the electron takes to spiral in from $r = a_B$ to $r = 0$. (For a more realistic estimate, see Problem 11.15.)

- 11.7 •• Many particle accelerators, including the cyclotron and the synchrotron (Sections 17.11 and 18.11), hold charged particles in a circular orbit using a suitable magnetic field. The centripetal acceleration, $a = v^2/r$, can be very large and can lead to serious energy loss by radiation, in accordance with Eq. (11.1). **(a)** Consider a 10-MeV proton in a cyclotron of radius 0.5 m. Use the formula (11.1) to calculate the rate of energy loss in eV/s due to radiation. **(b)** Suppose that we tried to produce electrons with the same kinetic energy in a circular machine of the same radius. In this case the motion would be relativistic and formula (11.1) is modified by an extra factor* of γ^4 :

$$P = \frac{2kq^2a^2\gamma^4}{3c^3} \quad (11.50)$$

Find the rate of energy loss of the electron, and compare with that for a proton. (Your answer for the electron should be enormously larger than for the proton. This explains why most electron accelerators are linear, not circular, since the acceleration in a linear accelerator — once $v \sim c$ — is far smaller than the centripetal acceleration considered here.)

- 11.8 •• Answer the same questions as in Problem 11.7, but assume that both the proton and electron have kinetic energy 10 GeV and move in a circle of radius 20 m. [In this case both particles are relativistic and you must use the relativistic formula (11.50).]

- 11.9 •• The ring called PEP-II at Stanford in California stores electrons orbiting around a circle[†] of radius 170 m. Because of the centripetal acceleration, $a = v^2/r$, the particles lose energy in accordance with Eq. (11.50) [which is the appropriate relativistic form of Eq. (11.1)]. **(a)** Find the rate of energy loss of a single 9-GeV electron. **(b)** If a total of 2×10^{12} electrons are radiating at this rate, what is the total power, in watts, needed to keep them at 9 GeV? (For comparison, the power used by a typical household appliance is on the order of 100 W.)

*Note that this is for the case of circular motion. For linear motion the factor is γ^6 .

[†]The actual device contains both curved and straight sections but is reasonably described as a single circle for the purposes of this problem.

- 11.10 ••** Figure 11.21 shows an electric field line from a charge q that was kicked abruptly from rest and has moved a distance d . The kink in the field line occurs between r and $r + \Delta r$, where $r \approx ct$ and t is the time since q was kicked. The radiation from q is carried by the transverse component of \mathcal{E} , shown as \mathcal{E}_{tr} in the picture. Prove that as a function of θ , \mathcal{E}_{tr} is proportional to $\sin \theta$. (This means that the maximum radiation is at $\theta = 90^\circ$ and there is no radiation at $\theta = 0$. For this reason, broadcasting antennas are oriented at 90° to the direction in which they need to transmit.)

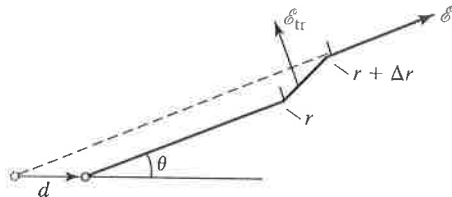


FIGURE 11.21

(Problem 11.10)

- 11.11 •• (a)** A classical point charge q of mass m is in a circular orbit of radius r around a fixed charge Q (with q and Q of opposite sign, of course). Starting from Eq. (11.1), derive a formula for the radiated power P in terms of q , m , r , and Q . **(b)** By what factor is P changed if we double q (leaving m , r , and Q unchanged)? **(c)** What if we double Q (with m , r , and q unchanged)?
- 11.12 ••** One of the difficulties with classical models of the atom was that they failed to predict the correct frequency for the radiation emitted. According to classical electromagnetic theory, the frequency of emitted radiation should equal the frequency of the orbiting electron. **(a)** Calculate the orbital frequencies, $f_{orb}(1)$ and $f_{orb}(2)$, of a classical electron in the $n = 1$ and $n = 2$ Bohr orbits of a hydrogen atom. **(b)** Now find the frequency $f_\gamma(2 \rightarrow 1)$ of the actual photon emitted in the $2 \rightarrow 1$ transition. Show that $f_\gamma(2 \rightarrow 1)$ is *not* equal to either $f_{orb}(2)$ or $f_{orb}(1)$ (or their average or their difference). **(c)** It turns out, however, that as $n \rightarrow \infty$ the orbital frequency $f_{orb}(n)$ of the n th orbit does approximate the frequency $f_\gamma(n \rightarrow n - 1)$ of a photon emitted in the transition ($n \rightarrow n - 1$). Prove that $f_{orb}(n) = E_R/(\pi\hbar n^3)$. Derive an expression for $f_\gamma(n \rightarrow n - 1)$, and show that it approaches $f_{orb}(n)$ as $n \rightarrow \infty$. (This result — called the **correspondence principle** — played an important role in the development of the Bohr model.)
- 11.13 ••** Consider the electron in a classical He^+ ion. Using the method of Example 11.1, find the radiated power predicted by the classical radiation formula (11.1) for a radius equal to that of the first Bohr orbit of He^+ . Compare your result with that predicted for hydrogen.
- 11.14 •••** Using the method of Problem 11.6, estimate the lifetime of a classical He^+ ion starting in the $n = 1$ Bohr orbit for He. Compare with the answer for hydrogen.
- 11.15 •••** Problem 11.6 estimates the time Δt for collapse of a classical hydrogen atom, making the approximation that it shrinks at a constant rate. Calculate Δt

without making that approximation, as follows: Use Eq. (11.49) to find dr/dt , the rate at which the orbit shrinks, as a function of r , and find the time for this classical atom to collapse entirely, by evaluating

$$\Delta t = \int_{a_B}^0 \frac{dr}{dr/dt}$$

SECTION 11.4 (More Quantum Formalism*)

- 11.16 •** The Hamiltonian operator is often described as a **linear operator** because it has the linear property that $H[A\psi(x) + B\phi(x)] = AH\psi(x) + BH\phi(x)$ for any two functions $\psi(x)$ and $\phi(x)$ and any two constant numbers A and B . Prove that the one-dimensional Hamiltonian (11.11) is linear.
- 11.17 ••** Show that the wave functions (11.21) for the infinite square well satisfy the orthogonality property that $\int_0^a \psi_m(x)^* \psi_n(x) dx = 0$ for $m \neq n$.

SECTION 11.5 (Transitions; Time-Dependent Perturbation Theory*)

- 11.18 • (a)** The first excited state of the sodium atom is 2.11 eV above the ground state. What wavelength radiation can cause transitions between these two levels? What sort of radiation is this? (visible, UV, etc.?) **(b)** Answer the same questions for the two $2p$ levels in hydrogen, which are shown in Fig. 9.7 and are 4.5×10^{-5} eV apart. (This is the fine-structure splitting discussed in Section 9.7.) **(c)** Answer the same questions for the lowest two levels of the ${}^7\text{Li}$ nucleus, which are 0.48 MeV apart.
- 11.19 •** The atoms of a certain monatomic gas have five energy levels: $E_1 = 0$, $E_2 = 5.4$, $E_3 = 8.2$, $E_4 = 8.6$, and $E_5 = 12.4$, all measured up from E_1 in eV. **(a)** If infrared light with wavelengths in the range from 3000 to 3200 nm shines through the gas, what transitions could it cause? If the gas was so cool that all atoms were in the ground state, would you expect to observe these transitions? **(b)** Answer the same questions for UV with wavelengths from 95 to 105 nm. **(c)** What wavelength light could de-excite the atoms from level 3 to level 2?
- 11.20 •** The wave functions for an infinite square well, with edges at $x = 0$ and $x = a$, are given in (11.21). Prove that if we move the origin to the middle of the well, these same functions are as given in (11.34) and (11.35). (Actually, some of the new functions differ by an unimportant sign from the original ones.)
- 11.21 ••** Confirm the result (11.37) for the probability $P(1 \rightarrow 2)$ for a transition from level 1 to level 2 of the rigid box, by evaluating the integral (11.36).
- 11.22 ••** Consider an electron that is initially in the first excited state of the infinite square well of Example 11.4 (Sec. 11.5). **(a)** If an electric field \mathcal{E} is switched on in the x direction, find the probability $P(2 \rightarrow 1)$ that the electron will be found in the ground state a short time Δt later. **(b)** Similarly, find the probability $P(2 \rightarrow 3)$ for excitation to the second excited state. **(c)** What are $P(2 \rightarrow 4)$ and $P(2 \rightarrow 5)$? Compare these various probabilities.

SECTION 11.8 (Atomic Selection Rules)

11.23 • When a quantum wave function Ψ is complex (with both real and imaginary parts), its probability density is $|\Psi|^2$, where $|\Psi|$ is the absolute value of Ψ , defined by Eq. (6.12) as $|\Psi| = \sqrt{\Psi_{\text{real}}^2 + \Psi_{\text{imag}}^2}$. Prove that $|\Psi|^2 = \Psi^*\Psi$, where Ψ^* is the complex conjugate of Ψ . (The complex conjugate z^* of any complex number $z = x + iy$, where x and y are real, is defined as $z^* = x - iy$.)

11.24 • The outermost (valence) electron of sodium is in a $3s$ state when the atom is in its ground state (Table 10.2 in Section 10.7). The valence electron can be excited to higher levels, the first few of which are shown in Fig. 11.22. Given the selection rule that only those transitions for which $\Delta l = \pm 1$ are allowed, indicate on this energy-level diagram all allowed transitions among the levels shown.

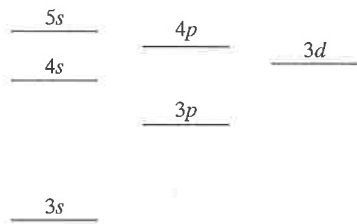


FIGURE 11.22

Some low-lying levels of any one electron in sodium (Problem 11.24).

11.25 • Figure 11.23 shows some of the lowest energy levels of the He atom. They are labeled by their configuration (for example, $1s2p$ means that the atom has one electron in the $1s$ level and one in the $2p$ level). The energy depends somewhat on the orientation of the two electrons' spins: If the spins are antiparallel, the total spin is zero (quantum number $s_{\text{tot}} = 0$); if the spins are parallel, the total spin has $s_{\text{tot}} = 1$. For a given configuration, the state with $s_{\text{tot}} = 1$ has slightly lower energy. (a) Explain why the $1s^2$ configuration has only $s_{\text{tot}} = 0$. (b) There is a selection rule $\Delta s_{\text{tot}} = 0$, that is, transitions in which s_{tot} changes are forbidden. Indicate all allowed transitions on the energy-level diagram. (Don't forget the selection rule $\Delta l = \pm 1$.) (c) Which excited levels would you expect to be metastable?

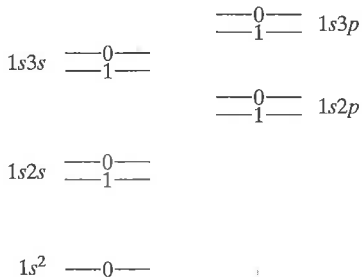


FIGURE 11.23

Some low levels of helium. (These are the energy levels of the whole atom, not the individual electrons.) The numbers 0 and 1 indicate the quantum number s_{tot} of the total spin (Problem 11.25).

11.26 •• The general proof of the selection rule (11.46) is beyond our scope, but you can prove it in a few special cases. (a) It is a fact that the wave function $\psi_{nlm}(\mathbf{r}) = R_{nl}(r)\Theta_{lm}(\theta)e^{im\phi}$ satisfies

$$\psi_{nlm}(-\mathbf{r}) = (-1)^l \psi_{nlm}(\mathbf{r}) \quad (11.51)$$

[This property is often described by saying that the wave function has **parity** $(-1)^l$.] Use the angular functions listed in Table 8.1 to prove (11.51) for all wave functions with $l = 0, 1$, or 2 . (b) The probability of a radiative transition $(n, l, m \rightarrow n', l', m')$ is given by (11.45), which now takes the form

$$P(n, l, m \rightarrow n', l', m') \propto \mathcal{E}_0^2 \left| \int \psi_{n'l'm'}^* x \psi_{nlm} dV \right|^2 \quad (11.52)$$

(This is for radiation polarized with \mathcal{E} in the x direction. For isotropic unpolarized radiation, we must average over this and the two corresponding expressions with x replaced by y and by z .) Use the property (11.51) to prove that the transition probability (11.52) is zero if $l' = l$ (whether the integrand is x, y , or z). This proves that transitions with $\Delta l = 0$ are forbidden — a particular case of (11.46).

11.27 •• There is a selection rule on the total angular momentum (all spins plus all orbital momenta) of an atom or nucleus. The total angular momentum is given by a quantum number j_{tot} , and it is found that transitions which do not satisfy $\Delta j_{\text{tot}} = 0$ or ± 1 are forbidden (that is, very improbable). Further, among the forbidden transitions the larger the value of Δj_{tot} , the more improbable a transition is. This trend applies to β decay in nuclei as well as to radiative transitions. Use these facts to explain why an excited state of the ^{180}Ta nucleus occurs naturally in measurable amounts. (See Appendix D.)

11.28 •• Consider two energy levels of the helium atom, in both of which the two electrons' spins are antiparallel (so that the total spin is zero, and the spins can be ignored) and one of the electrons is in the lowest ($1s$) orbital. In the upper level the second electron has $l = 2$; in the lower level the second electron has $l = 1$. The atom is placed in a magnetic field, and (as described in Section 9.4) the upper level splits into five equally spaced sublevels and the lower into three sublevels (with the same spacing). (a) Sketch the resulting levels. (b) There are, in principle, 15 different possible transitions from the upper ($l = 2$) level to the lower ($l = 1$) level. Show that because the sublevels all have the same spacing, there are actually only seven distinct energy differences. (c) The selection rules for these transitions are $\Delta l = \pm 1$ and $\Delta m = 0$ or ± 1 ; that is, only transitions that satisfy these rules are allowed. Indicate all of the allowed transitions on your energy level diagram. (d) How many distinct photon frequencies will result from allowed transitions between the two levels? This is the normal Zeeman effect described in Section 9.4.

11.29 •• Carry out the same tasks as in Problem 11.28 (with the numbers modified where necessary), using the same upper level ($l = 2$) but with a lower level in which one electron is in an s state and the other in an f state ($l = 3$).