Jacob Wachtman

Intro to Quantum Computing : ECES 622 - Winter 2007

Monday 5:00pm

Homework 3, Due date: Friday ~~February~~ 9th, 2007 by 5p.m in my office
If I am not there, slip your homework under the door.

1. (30 pts) Write the outer product decomposition of the Pauli matrix σ_x using the unit vectors $|v_1>$ and $|v_2>$ in C^2 given below as a basis,

$$|v_1>= \frac{1}{\sqrt{2}}[1,1]^{\dagger} \tag{1}$$

and

$$|v_2>= \frac{1}{\sqrt{2}}[1,-1]^{\dagger} \tag{2}$$

2. (30 pts) The Hadamard matrix will be a gate we will encounter in many quantum circuits later on. It is given by

$$H = \frac{1}{\sqrt{2}}[(|0>+|1>)<0|+(|0>-|1>)<1|] \tag{3}$$

(a) Calculate the explicit form of the 2x2 matrix for H.

(b) Calculate the eigenvalues and eigenvectors of H.

(c) Calculate the Kronecker product $H \otimes H$ by expanding the product starting with the expression for H given above using the rules for the Kronecker product. Then, write the final answer as a 4x4 matrix. Redo the problem while starting with the 2X2 form of the H matrix and doing the Kronecker product using the matrix representation of operators as we described in the notes. The two final answers should agree of course.

3. (30 pts) Problem 2.1 page 117 in the book.

4. (30 pts) Consider the following two unit vectors in C^2

$$|v_1>= \frac{1}{\sqrt{2}}[1,1]^{\dagger} \tag{4}$$

and

$$|v_2>= \frac{1}{\sqrt{2}}[1,-1]^{\dagger} \tag{5}$$

where the $\dagger$ stands for the conjugate followed by transpose operation. Calculate the tensor product $|v_1>\otimes|v_1>\otimes|v_2>$ and express your answer as a 8x1 column vector.

5. (30 pts) Consider the following two Bell states representing a system of two qubits in $C_2 X C_2$:

$$|\beta_{11}>= \frac{|00>+|11>}{\sqrt{2}} \tag{6}$$

and

$$|\beta_{01}>= \frac{|01>+|10>}{\sqrt{2}} \tag{7}$$

These qubits are vectors (kets) of C_4 (4x1 column vectors).

- What is the explicit form of the two bras (1X4 row vectors) associated to these two (4x1) kets?
- Show explictly (i.e, give all intermediate steps) that $|\beta_{01}>$ is orthogonal to $|\beta_{11}>$.
- What is the average value of the tensor product

$$\sigma_y \otimes \sigma_z \tag{8}$$

(where the σ's are the Pauli matrices) in the 2-qubit state

$$[\frac{1}{2}(|0>-|1>)] \otimes [\frac{1}{2}(|0>+|1>)]? \tag{9}$$

6. (30 pts) Consider the following qubit

$$|\psi>= \frac{1}{\sqrt{5}}(2|0>-|1>) \tag{10}$$

Calculate the variances of σ_x and σ_y and show that their product satisfy the generalyzed Heisenberg relation derived in class.

Calculate the angles θ and ϕ associated to the spinor given above to be able to show its location on the Bloch sphere.

7. (30 pts) As derived in class, the most general form of a spinor representing the state of a qubit on the Bloch sphere is given by

$$|\xi_n^+>= cos\frac{\theta}{2}|0> + sin\frac{\theta}{2}e^{i\phi}|1>= \begin{bmatrix} cos\frac{\theta}{2} \\ sin\frac{\theta}{2}e^{i\phi} \end{bmatrix} \tag{11}$$

which is eigenstate of $\vec{\sigma}.\vec{n}$ with eigenvalue +1.

We also show that

$$|\xi_n^->= sin\frac{\theta}{2}|0> - cos\frac{\theta}{2}e^{i\phi}|1>= \begin{bmatrix} sin\frac{\theta}{2} \\ -cos\frac{\theta}{2}e^{i\phi} \end{bmatrix} \tag{12}$$

is eigenstate of $\vec{\sigma}.\vec{n}$ with eigenvalue -1.

For an electron characterized by the spinor $|\xi_n^+>$ above, show that the average values of the Pauli spin matrices are given by

$$<\xi_n^+|\sigma_x|\xi_n{}^+>= sin\theta cos\phi \tag{13}$$

$$<\xi_n^+|\sigma_y|\xi_n{}^+>= sin\theta sin\phi \tag{14}$$

$$<\xi_n^+|\sigma_z|\xi_n{}^+>= cos\theta \tag{15}$$

which are exactly the carthesian components of the unit vector $\hat{n}$ giving the location of the spinor on the Bloch sphere.

Similarly, starting with $|\xi_n^->$, show that

$$<\xi_n^-|\sigma_x|\xi_n{}^->= -sin\theta cos\phi \tag{16}$$

$$<\xi_n^-|\sigma_y|\xi_n{}^->= -sin\theta sin\phi \tag{17}$$

$$<\xi_n^-|\sigma_z|\xi_n{}^->= -cos\theta \tag{18}$$

which are the carthesian coordinates of a unit vector diametrically opposite to the previous unit vector $\hat{n}$.

$$\boxed{\begin{array}{c}\underline{\text{Homework \#3 - Solutions}}\\ \underline{\text{Winter 2007}}\end{array}}$$

#1 $|v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ $\quad |v_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$$\sigma_x = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$|0\rangle = \frac{1}{\sqrt{2}}\left[|v_1\rangle + |v_2\rangle\right] \qquad |1\rangle = \frac{1}{\sqrt{2}}\left[|v_1\rangle - |v_2\rangle\right]$$

$$\Rightarrow \sigma_x = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(|v_1\rangle + |v_2\rangle\right)\left(\langle v_1| - \langle v_2|\right)$$
$$+ \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(|v_1\rangle - |v_2\rangle\right)\left(\langle v_1| + \langle v_2|\right)$$

$$\sigma_x = \frac{1}{2}\left(|v_1\rangle\langle v_1| - |v_1\rangle\langle v_2| + \cancel{|v_2\rangle\langle v_1|} - |v_2\rangle\langle v_2|\right)$$
$$+ \frac{1}{2}\left(|v_1\rangle\langle v_1| + |v_1\rangle\langle v_2| - \cancel{|v_2\rangle\langle v_1|} - |v_2\rangle\langle v_2|\right)$$

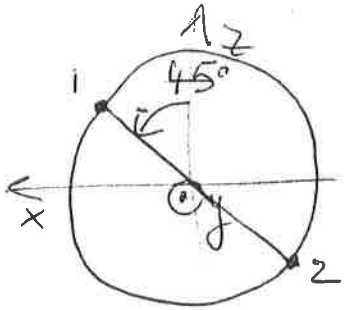
So $\boxed{\sigma_x = |v_1\rangle\langle v_1| - |v_2\rangle\langle v_2|}$

2a $$H = \frac{1}{\sqrt{2}}\Big[\underbrace{|0\rangle\langle 0|}_{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}} + \underbrace{|1\rangle\langle 0|}_{\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}} + \underbrace{|0\rangle\langle 1|}_{\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}} - \underbrace{|1\rangle\langle 1|}_{\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}}\Big]$$

$$\Rightarrow H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

(2b) $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

<u>Eigenvectors</u> As seen in class



$\varphi = 0$

$|\psi_1\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle$ & $|\psi_2\rangle = \cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|0\rangle$

are eigenvectors of H

<u>Corresponding eigenvalue</u>

$$|\psi_1\rangle \rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{8} \\ \sin\frac{\pi}{8} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos\frac{\pi}{8} + \sin\frac{\pi}{8} \\ \cos\frac{\pi}{8} - \sin\frac{\pi}{8} \end{bmatrix}$$

$$\begin{bmatrix} \cos\frac{\pi}{4}\cos\frac{\pi}{8} + \sin\frac{\pi}{4}\sin\frac{\pi}{8} \\ \sin\frac{\pi}{4}\cos\frac{\pi}{8} - \cos\frac{\pi}{4}\sin\frac{\pi}{8} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{8} \\ \sin\frac{\pi}{8} \end{bmatrix} = +1\begin{bmatrix} \cos\frac{\pi}{8} \\ \sin\frac{\pi}{8} \end{bmatrix}$$

So $|\psi_1\rangle$ has a corresponding eigenvalue $+1$

Similarly for $|\psi_2\rangle = \begin{bmatrix} \cos\pi/8 \\ -\sin\pi/8 \end{bmatrix}$, the eigenvalue is -1

2(c)

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & +1\end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|\right]$$

$$H \otimes H = \frac{1}{2}\left[|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|\right] \otimes \left[|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|\right]$$

So

$$H^{\otimes 2} = \frac{1}{2}\Big[|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 0| + |0\rangle\langle 0| \otimes |0\rangle\langle 1| - |0\rangle\langle 0| \otimes |1\rangle\langle 1|$$

+ all other 12 terms

$$H^{\otimes 2} = \frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \cdots\right.$$ Total of $\underline{\underline{16}}$ 4×4 matrices

Each brings one at a time one element of $H^{\otimes 2}$

It is either +1 or −1

$\longrightarrow$ $\boxed{\text{2 ways of getting } H^{\otimes 2}}$

③ Prove that

$$f(\theta \vec{n}\cdot\vec{\sigma}) = \left[\frac{f(\theta)+f(-\theta)}{2}\right] \mathbb{I} + \left[\frac{f(\theta)\ f(-\theta)}{2}\right] \vec{\sigma}\cdot\vec{n}$$

which is a generalization of

$$e^{i\theta\vec{\sigma}\cdot\vec{n}} = \cos\theta\ \mathbb{I} + i\sin\theta\,(\vec{\sigma}\cdot\vec{n})$$

we will use the fact that $(\vec{\sigma}\cdot\vec{n})^2 = \mathbb{I}$

If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$ is the Taylor expansion of $f(x)$

$$f(x) = \underbrace{\sum_{n=0}^{\text{even}} \frac{x^n}{n!} f^{(n)}(0)}_{g(x)} + \underbrace{\sum_{n=0}^{\text{odd}} \frac{x^n}{n!} f^{(n)}(0)}_{h(x)}$$

$$g(x) = \frac{f(x)+f(-x)}{2} \qquad h(x) = \frac{f(x)-f(-x)}{2}$$

$$f(\theta\vec{n}\cdot\vec{\sigma}) = \underbrace{\sum_{n}^{\text{even}} \frac{\theta^n}{n!} \underbrace{(\vec{\sigma}\cdot\vec{n})^n}_{\mathbb{I}} f^{(n)}(0)}_{g(\theta)} + \underbrace{\sum_{n}^{\text{odd}} \frac{\theta^n}{n!} \underbrace{(\vec{\sigma}\cdot\vec{n})^n}_{=\vec{\sigma}\cdot\vec{n}} f^{(n)}(0)}_{\underbrace{\left[\sum_{n}^{\text{odd}} \frac{\theta^n}{n!} f^{(n)}(0)\right]}_{h(\theta)} \vec{\sigma}\cdot\vec{n}}$$

$$g(\theta) = \frac{f(\theta)+f(-\theta)}{2} \ ; \ h(\theta) = \frac{f(\theta)-f(-\theta)}{2}$$

so

$$\boxed{f(\theta\vec{n}\cdot\vec{\sigma}) = \left(\frac{f(\theta)+f(-\theta)}{2}\right) \mathbb{I} + \left[\frac{f(\theta)-f(-\theta)}{2}\right] \vec{\sigma}\cdot\vec{n}}$$

#4 $|v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ $\quad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$$|v_1\rangle \otimes |v_1\rangle = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

$$|v_1\rangle \otimes |v_1\rangle \otimes |v_2\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\-1\end{bmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{bmatrix}1\\-1\\1\\-1\\1\\-1\\1\\-1\end{bmatrix}$$

#5 (a) $|B_{11}\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

$|B_{01}\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$

$|00\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$; $|11\rangle = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$|01\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$; $|10\rangle = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$

So $|B_{11}\rangle = \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$; $|B_{01}\rangle = \dfrac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$

(b) $\langle B_{11}|B_{01}\rangle = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = 0$ indeed

(5c) Consider the tensor product

$$\sigma_y \otimes \sigma_z -$$

Calculate its average value in the state

$$|\psi\rangle_{2bit} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left[(\langle 0| - \langle 1|)\,\sigma_y(|0\rangle - |1\rangle)\right]\left[(\langle 0| + \langle 1|)\,\sigma_z(|0\rangle + |1\rangle)\right]$$

$$= \frac{1}{2}\left[\underbrace{\langle 0|\sigma_y|0\rangle}_{0} - \underbrace{\langle 0|\sigma_y|1\rangle}_{-i} - \underbrace{\langle 1|\sigma_y|0\rangle}_{+i} + \underbrace{\langle 1|\sigma_y|1\rangle}_{0}\right]$$

$$\left[\underbrace{\langle 0|\sigma_z|0\rangle}_{+1} + \underbrace{\langle 0|\sigma_z|1\rangle}_{0} + \underbrace{\langle 1|\sigma_z|0\rangle}_{0} + \underbrace{\langle 1|\sigma_z|1\rangle}_{-1}\right]$$

$$\rightarrow \boxed{{}_{2bit}\langle\psi|\,\sigma_y \otimes \sigma_z\,|\psi\rangle_{2bit} = 0}$$

#6

$$|\psi\rangle = \frac{1}{\sqrt{5}}\left[2|0\rangle - |1\rangle\right]$$

is normalized since $\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{-1}{\sqrt{5}}\right)^2 = 1$

(c) $$\overline{\sigma_x} = \langle\psi|\sigma_x|\psi\rangle = \frac{1}{5}\left[2\langle 0| - \langle 1|\right]\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left[2|0\rangle - |1\rangle\right]$$

bit flip

$$\overline{\sigma_x} = \frac{1}{5}\left[\left(2\langle 0| - \langle 1|\right)\left(2|1\rangle - |0\rangle\right)\right]$$

So $$\overline{\sigma_x} = \frac{1}{5}\left[-2-2\right] = \frac{-4}{5}$$

$$\overline{\sigma_y} = \langle\psi|\sigma_y|\psi\rangle = \frac{1}{5}\left[\left(2\langle 0| - \langle 1|\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\left(2|0\rangle - |1\rangle\right)\right]$$

$$= \frac{1}{5}(2,-1)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{5}(2,-1)\begin{pmatrix} i \\ 2i \end{pmatrix} = \frac{i}{5}(2-2) = 0$$

$$\Delta\sigma_x = \sqrt{\langle\sigma_x^2\rangle - \langle\sigma_x\rangle^2} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\Delta\sigma_y = \sqrt{\langle\sigma_y^2\rangle - \langle\sigma_y\rangle^2} = 1$$

$$\Delta\sigma_x\Delta\sigma_y \geq \frac{|\langle\psi|[\sigma_x,\sigma_y]|\psi\rangle|}{2} = |\langle\psi|\sigma_z|\psi\rangle|$$

$$[\sigma_x, \sigma_y] = 2i\,\sigma_z$$

$$\overline{\sigma_z} = \frac{1}{5}\left[\left(2\langle 0| - \langle 1|\right)\sigma_z\left(2|0\rangle - |1\rangle\right)\right]$$

$$\overline{\sigma_z} = \frac{1}{5}(2,-1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{5}(2,-1)\begin{pmatrix} 2 \\ +1 \end{pmatrix} = \frac{3}{5}$$

so, in this case, $\boxed{\Delta\sigma_x\,\Delta\sigma_y = 3/5}$

6 b θ, ϕ for

$$|\psi\rangle = -\frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle$$

$$= \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\varphi}|1\rangle$$

$\Rightarrow \varphi = \pi$

$$\cos\frac{\theta}{2} = \frac{2}{\sqrt{5}}$$

$$\rightarrow \theta = 2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) = 53^{\circ}.13$$

#7

$$|\xi_n^+\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{bmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle \xi_n^+ | \sigma_x | \xi_n^+ \rangle = \left(\cos\frac{\theta}{2} , \sin\frac{\theta}{2} e^{-i\phi}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

$$= \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2} e^{-i\phi}\right) \begin{pmatrix} \sin\frac{\theta}{2} e^{i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

$$= \cos\frac{\theta}{2} \sin\frac{\theta}{2} \underbrace{\left(e^{i\phi} + e^{-i\phi}\right)}_{2\cos\phi} = \cos\phi \sin\theta$$

using the fact that $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

Similarly $\langle \xi_n^- | \sigma_x | \xi_n^- \rangle = -\cos\phi \sin\theta$

The other relations can be derived following a similar procedure.

Problem IV:

Consider the qubit of a system characterized by the following spinor at time t = 0,

$$|\psi(t=0)\rangle = \sqrt{\frac{2}{3}}|0\rangle - \frac{1}{\sqrt{3}}|1\rangle \quad (11)$$

In a quantum circuit, this qubit is submitted to the action of three gates equivalent to the application of the following product of 3 matrices

$$XHX \quad (12)$$

where X and H are the Pauli and Hadamard matrices, respectively.

- Calculate the explicit form of the qubit at time t_1, i.e, after application of the product X H X.
- At this time t_1, if you measure the x component of $S_x = \frac{\hbar}{2}\sigma_x$, what is the probability to measure $+\frac{\hbar}{2}$?
- If you indeed measured $+\frac{\hbar}{2}$ after measuring S_x in the previous step What is the expression of the qubit right after making the measurement? Do not forget to normalize the expression. (c) $\frac{-1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$
- If you let the qubit after your measurement at t_1 evolve again and it is submitted to the action of the another set of 3 gates,

$$XHZ \quad (13)$$

after which you decide to measure the z-component $S_z = \frac{\hbar}{2}\sigma_z$

Calculate the standard deviation of the measurement of $S_z = \frac{\hbar}{2}\sigma_z$ on the final qubit.

Remember: the Hadamard matrix is given by

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (14)$$

$$|\psi(0)\rangle = \begin{pmatrix}\sqrt{\frac{2}{3}} \\ -1/\sqrt{3}\end{pmatrix} \text{ is normalized}$$

$$XHX = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{-1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix} = \begin{pmatrix}\frac{-1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}$$

$$(a)\ |\psi(t_1)\rangle = \begin{pmatrix}\frac{-1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}|\psi(0)\rangle = \begin{bmatrix}-(1+\sqrt2)/\sqrt6\\(\sqrt2-1)/\sqrt6\end{bmatrix}$$

$$(b)\ P(+\tfrac{\hbar}{2}) = |\langle +|\psi(t_1)\rangle|^2 = \left|\frac{1}{\sqrt2}(1,1)\begin{bmatrix}-(\sqrt2+1)/\sqrt6\\(\sqrt2-1)/\sqrt6\end{bmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\left[-\frac{2}{\sqrt6}\right]\right|^2$$

$$P(+\tfrac{\hbar}{2}) = \left|\frac{1}{\sqrt3}\right|^2 = \frac{1}{3} = \boxed{}$$

$$|\psi(0)\rangle = \begin{bmatrix} \sqrt{\frac{2}{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix} = \sqrt{\frac{2}{3}}|0\rangle - \frac{1}{\sqrt{3}}|1\rangle$$

$$Z H \underbrace{X^2}_{\mathbb{I}} H X = Z \underbrace{H^2}_{\mathbb{I}} X = Z X$$

$$|\psi(t_2)\rangle = Z X |\psi(0)\rangle = Z\left[\sqrt{\frac{2}{3}}|1\rangle - \frac{1}{\sqrt{3}}|0\rangle\right]$$

$$= -\sqrt{\frac{2}{3}}|1\rangle - \frac{1}{\sqrt{3}}|0\rangle$$

$$\langle\psi(t_2)|S_z|\psi(t_2)\rangle = \frac{\hbar}{2}\left(\frac{-1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix}$$

$$= \frac{\hbar}{2}\left[\frac{1}{3} - \frac{2}{3}\right] = -\frac{\hbar}{6}$$

$$\langle\psi(t_2)|S_z^2|\psi(t_2)\rangle = \frac{\hbar^2}{4}$$

$$\sigma_z^2 = \frac{\hbar^2}{4} - \frac{\hbar^2}{36} = \frac{8}{36}\hbar^2 = \frac{2}{9}\hbar^2$$

$$\Longrightarrow \boxed{\sigma_z = \frac{\sqrt{2}}{3}\hbar}$$

(IV) Give the (4×1) column vector obtained as the following tensor product expression

$$(H \otimes \sigma_x)(|0\rangle \otimes |1\rangle)$$

$$\|$$

$$H|0\rangle \otimes \sigma_x|1\rangle$$

$$\|$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$\|$$

$$\frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Keerthana Kalyanasundaram.
301-06-8165

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$30 + 29 + 30 + 15 + 20 = 124/140$

Dr. Caliay

Midterm Exam: February 10, 2004

- (I) Starting with the following normalized kets

$$|a>=\sqrt{\frac{3}{4}}|0>+\sqrt{\frac{1}{4}}|1> \quad (1)$$

and

$$|b>=\sqrt{\frac{3}{4}}|0>-\sqrt{\frac{1}{4}}|1> \quad (2)$$

Consider the operator ρ defined as follows

$$\rho=\frac{1}{2}|a><a|+\frac{1}{2}|b><b| \quad (3)$$

(a) Show that

$$\rho=\frac{3}{4}|0><0|+\frac{1}{4}|1><1| \quad (4)$$

(b) Then, show that

$$Tr(\rho)=1. \quad (5)$$

(c) Calculate $Tr(\rho^2)$. Prove that the latter is less than 1.
In (b) and (c) above, Tr stands for the trace of the corresponding operators.

(a) $\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|$

$$= \frac{1}{2}\left(\sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle\right)\left(\sqrt{\frac{3}{4}}\langle 0| + \sqrt{\frac{1}{4}}\langle 1|\right) +$$

$$\frac{1}{2}\left(\sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle\right)\left(\sqrt{\frac{3}{4}}\langle 0| - \sqrt{\frac{1}{4}}\langle 1|\right)$$

$|0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

First Term:

$$= \frac{1}{2}\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right)$$

PTO

Second Term:-

$$= \frac{1}{2}\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right)$$

First term + Second term.

$$\Rightarrow$$

$$= \frac{1}{2}\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right) + \left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right)$$

$$\rho = \cancel{\frac{2}{2}}\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}(|1\rangle\langle 1|)\right) = \underline{\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right)}$$ Hence the proof.

(b) $\underline{Tr(\rho) = 1}$

$$\rho = \left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}(|1\rangle\langle 1|)\right) = \left(\frac{3}{4}\begin{pmatrix}1\\0\end{pmatrix}(1\ 0) + \frac{1}{4}\begin{pmatrix}0\\1\end{pmatrix}(0\ 1)\right)$$

$$= \frac{3}{4}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}3/4 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1/4\end{pmatrix}$$

$$= \begin{pmatrix}3/4 & 0\\0 & 1/4\end{pmatrix}$$ ✓

$$Tr(\rho) = \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1 //$$ Hence the Proof ✓

c) $Tr(\rho^2) < 1$.

$$\rho^2 = \rho \times \rho = \begin{pmatrix}3/4 & 0\\0 & 1/4\end{pmatrix}\begin{pmatrix}3/4 & 0\\0 & 1/4\end{pmatrix} = \begin{pmatrix}9/16 & 0\\0 & 1/16\end{pmatrix}$$

$$(\rho^2) = \frac{9}{16} + \frac{1}{16} = \frac{10}{16} = \frac{5}{8} < 1 \quad \therefore \boxed{Tr(\rho^2) < 1}$$

Hence the Proof.

- (IIa) Write the outer product representation of the Hadamard matrix H using the orthonormal basis $|0>, |1>$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Outer Product Representation of $H \Rightarrow \sum_{ij=0}^{1} \langle i|H|j\rangle\, |i\rangle\langle j|$

$\therefore$ Basis $|0\rangle$ & $|1\rangle$

$$\langle 0|H|1\rangle = (1\ 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(1\ 0)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}$$

$$\langle 0|H|0\rangle = (1\ 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(1\ 0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}$$

$$\langle 1|H|0\rangle = (0\ 1)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(0\ 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}$$

$$\langle 1|H|1\rangle = (0\ 1)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(0\ 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-1) = \frac{-1}{\sqrt{2}}$$

$$\therefore H = \frac{1}{\sqrt{2}}|0\rangle\langle 1| + \frac{1}{\sqrt{2}}|0\rangle\langle 0| + \frac{1}{\sqrt{2}}|1\rangle\langle 0| - \frac{1}{\sqrt{2}}|1\rangle\langle 1|$$

$$\boxed{H = \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right]}$$

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Outer Product representation

- (IIb) Use the result of the previous exercise to calculate the average value and standard deviation of the Hadamard operator for a qubit in the ket

$(1\ 0)\begin{pmatrix}1\\0\end{pmatrix} = 1$

$$|a> = \sqrt{\frac{3}{4}}|0> + \sqrt{\frac{1}{4}}|1>$$

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Average Value :-

$$\langle a|H|a\rangle = \left(\sqrt{3/4}\langle 0| + \sqrt{\tfrac{1}{4}}\langle 1|\right)\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}}_{H}\left(\sqrt{\tfrac{3}{4}}|0\rangle + \sqrt{\tfrac{1}{4}}|1\rangle\right)$$

$$= \left(\sqrt{\tfrac{3}{4}}\langle 0| + \sqrt{\tfrac{1}{4}}\langle 1|\right)\frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)\left(\sqrt{\tfrac{3}{4}}|0\rangle + \sqrt{\tfrac{1}{4}}|1\rangle\right)$$

$$= \frac{3}{4}\langle 0|H|0\rangle + \frac{1}{4}\langle 1|H|1\rangle$$

$$= \frac{3}{4}(1\ 0)\frac{1}{\sqrt{2}}\underset{2\times 2}{\begin{pmatrix}1&1\\1&-1\end{pmatrix}}\underset{2\times 1}{\begin{pmatrix}1\\0\end{pmatrix}} + \frac{1}{4}(0\ 1)\frac{1}{\sqrt{2}}\underset{2\times 2}{\begin{pmatrix}1&1\\1&-1\end{pmatrix}}\underset{2\times 1}{\begin{pmatrix}0\\1\end{pmatrix}}$$

$$= \frac{3}{4}\cdot\frac{1}{\sqrt{2}}(1\ 0)\underset{2\times 1}{\begin{pmatrix}1\\1\end{pmatrix}} + \frac{1}{4}\cdot\frac{1}{\sqrt{2}}(0\ 1)\begin{pmatrix}1\\-1\end{pmatrix}$$

$\sqrt{8} = \sqrt{2\times 2\times 2} = 2\sqrt{2}$

$$= \frac{1}{\sqrt{2}}\left\{\frac{3}{4} - \frac{1}{4}\right\} = \frac{\cancel{2}}{\cancel{4}\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

Standard Deviation :-

$$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2} = \sqrt{\langle a|H^2|a\rangle - (\langle a|H|a\rangle)^2}$$

$$\langle a|H^2|a\rangle = \left(\sqrt{\tfrac{3}{4}}\langle 0| + \sqrt{\tfrac{1}{4}}\langle 1|\right)H^2\left(\sqrt{\tfrac{3}{4}}|0\rangle + \sqrt{\tfrac{1}{4}}|1\rangle\right)$$

But $H^2 = I$

$$= \frac{3}{4}\langle 0|H^2|0\rangle + \frac{1}{4}\langle 1|H^2|1\rangle = \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$$

$$\langle a|H|a\rangle^2 = \cancel{\left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{4\times 2}} = \frac{1}{8}$$

Please check Opposite Pg. PTO

$H|a\rangle$ $\langle a|$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)\left(\sqrt{\tfrac{3}{4}}\,|0\rangle + \sqrt{\tfrac{1}{4}}\,|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big\{\sqrt{\tfrac{3}{4}}\left(|0\rangle\right) + \sqrt{\tfrac{3}{4}}\,|0\rangle \cancel{\langle 1|0\rangle} + \sqrt{\tfrac{3}{4}}\,|1\rangle\langle 0|0\rangle^{=1} - \sqrt{\tfrac{3}{4}}\,|1\rangle\cancel{\langle 1|0\rangle}^{=0}$$

$$+ \sqrt{\tfrac{1}{4}}\,|0\rangle\cancel{\langle 0|1\rangle} + \sqrt{\tfrac{1}{4}}\,|0\rangle\langle 1|1\rangle^{=1} + \sqrt{\tfrac{1}{4}}\,|1\rangle\cancel{\langle 0|1\rangle}^{=0} - \sqrt{\tfrac{1}{4}}\,|1\rangle\cancel{\langle 1|1\rangle}^{=1}\Big\}$$

$$= \frac{1}{\sqrt{2}}\Big\{\sqrt{\tfrac{3}{4}}\,|0\rangle + \sqrt{\tfrac{3}{4}}\,|1\rangle + \sqrt{\tfrac{1}{4}}\,|0\rangle - \sqrt{\tfrac{1}{4}}\,|1\rangle\Big\} =$$

$$\langle a|H|a\rangle = \left(\sqrt{\tfrac{3}{4}}\,\langle 0| + \sqrt{\tfrac{1}{4}}\,\langle 1|\right)\frac{1}{\sqrt{2}}\Big\{\sqrt{\tfrac{3}{4}}\,|0\rangle + \sqrt{\tfrac{3}{4}}\,|1\rangle + \sqrt{\tfrac{1}{4}}\,|0\rangle - \sqrt{\tfrac{1}{4}}\,|1\rangle\Big\}$$

$$= \frac{1}{\sqrt{2}}\Big(\tfrac{3}{4}\langle 0|0\rangle + \tfrac{3}{4}\cancel{\langle 0|1\rangle} + \sqrt{\tfrac{3}{16}}\langle 0|0\rangle - \sqrt{\tfrac{3}{16}}\cancel{\langle 0|1\rangle} + \sqrt{\tfrac{3}{16}}\cancel{\langle 1|0\rangle} + \sqrt{\tfrac{3}{16}}\langle 1|1\rangle$$

$$+ \tfrac{1}{4}\cancel{\langle 1|0\rangle} - \tfrac{1}{4}\langle 1|1\rangle\Big\}$$

$(1\ 0)\begin{pmatrix}0\\1\end{pmatrix} = 0$

$$= \frac{1}{\sqrt{2}}\Big(\tfrac{3}{4} + \sqrt{\tfrac{3}{16}} + \sqrt{\tfrac{3}{16}} - \tfrac{1}{4}\Big\} = \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{\cancel{2}^{1}}{\cancel{4}\,2}\sqrt{\frac{3}{\cancel{16}}}\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \cancel{2}\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\tfrac{3}{4}} & \sqrt{\tfrac{1}{4}}\end{pmatrix}\begin{pmatrix}\sqrt{3/4} + \sqrt{\tfrac{1}{4}} \\ \sqrt{3/4} - \sqrt{\tfrac{1}{4}}\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\frac{3}{4} + \sqrt{\frac{3}{16}} + \sqrt{\frac{3}{16}} - \frac{1}{4}\right) =$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}} \rightarrow \text{Avg. Value}$$

$$\frac{8-4+2\sqrt{3}}{8} = \frac{4+2\sqrt{3}}{8}$$

SD :

$$\Delta(H) = \sqrt{\langle H^2\rangle - \langle H\rangle^2}$$

$$\langle\psi|H^2|\psi\rangle = 1 \quad , \quad \langle H\rangle^2 = \left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)^2 = \frac{(1+\sqrt{3})^2}{8} = \frac{1+3+2\sqrt{3}}{8}$$

$$\Delta H = \sqrt{1 - \frac{(4+2\sqrt{3})}{8}} = \sqrt{\frac{4+2\sqrt{3}}{8}} = \sqrt{\frac{\cancel{2}(2+\sqrt{3})}{\cancel{8}\,4}} = \sqrt{\frac{2+\sqrt{3}}{4}} \rightarrow \text{SD.}$$

Standard deviation:-

$$\Delta H = \sqrt{1 - \frac{1}{8}}$$

$$= \sqrt{\frac{8-1}{8}} = \sqrt{\frac{7}{8}} \;//$$

$$\therefore \quad E(H) = \frac{1}{2\sqrt{2}} \;//$$

$$\Delta (H) = \sqrt{\frac{7}{8}} \;//$$

- (IIIa) Calculate the tensor product $H \otimes H$ and give the result as a 4x4 matrix.

Tensor:- $H \otimes H$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

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$$= \frac{1}{2} \begin{bmatrix} 1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & 1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ 1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$H \otimes H = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} //$$

(4x4 matrix)

- (IIIb) Calculate explicitely (i.e, as a 4x1 column vector) the state of a 2-qubit system given by

$$(H \otimes H)(|0> \otimes |0>)$$

and show that the result is an equal superposition of all computational basis states in C^4. Remember that the basis states in C^4 which form an orthonormal basis are given by $|00>$, $|01>$, $|10>$, and $|11>$.

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$$(H \otimes H)(|0\rangle \otimes |0\rangle) = H|0\rangle \otimes H|0\rangle$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \rightarrow (1)$$

Let :-

$$|\phi_1\rangle \otimes |\phi_2\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$$

where, α_1, β_1, α_2 & β_2 are constants

$$\Rightarrow \left[\alpha_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \otimes \left[\alpha_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \\ \beta_1 \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{pmatrix} \rightarrow (2)$$

$\Rightarrow |0\rangle|0\rangle \Rightarrow |00\rangle$

$\Rightarrow |0\rangle|1\rangle \Rightarrow |01\rangle$

$\Rightarrow |1\rangle|0\rangle \Rightarrow |10\rangle$

$\Rightarrow |1\rangle|1\rangle \Rightarrow |11\rangle$

- When we compare, $\rightarrow$ (1) with (2)

$\alpha_1\alpha_2 = 1/2$; $\alpha_1\beta_2 = 1/2$; $\beta_1\alpha_2 = 1/2$, $\beta_1\beta_2 = 1/2$. Hence, we can see that it is all $\neq 0$ and so an equal $(1/2)$ superposition of all computational Basis in C^4.

- (IV) Consider the Bloch sphere on the attached figure and find out the values of α and β in the representation

$$|\psi> = \alpha|0> + \beta|1>$$

for the points 1 and 2 shown on the surface of the Bloch sphere.

- Calculate the values of the inner product of the two kets representing the points 1 and 2 and show that these two kets are NOT orthogonal.

- Give the location on the Bloch sphere of the two qubits $|\psi_1'>$ and $|\psi_2'>$ which are orthogonal to $|\psi_1>$ and $|\psi_2>$, respectively.

- Give the values of the (α, β) representing the components of the two qubits $|\psi_1'>$ and $|\psi_2'>$ along the kets $|0>$ and $|1>$.

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(a) $|\psi> = \cos\frac{\theta}{2}|0> + \sin\frac{\theta}{2}e^{i\phi}|1>$

Point 1:-

$\alpha = \cos\frac{\theta}{2} = \cos\frac{\theta}{2}$ $\quad\therefore\ \theta = \theta$

$\beta = \sin\frac{\theta}{2}e^{i\phi} = \sin\frac{\theta}{2}(\cos\phi + i\sin\phi)$ $\quad\therefore\ \theta = \theta, \phi = \phi$

Point 2:- $\theta = \theta - \pi$; $\phi = -\phi$ $\qquad e^{-i\phi} = \cos\phi - i\sin\phi$

$\alpha = \cos\frac{\theta}{2} = \cos\frac{(\theta-\pi)}{2}$

$\beta = \sin\frac{\theta}{2}e^{i\phi} = \sin\frac{(\theta-\pi)}{2}e^{-i\phi} =$

$|\psi_1> = \cos\frac{\theta}{2}|0> + \sin\frac{\theta}{2}e^{i\phi}|1> \checkmark = \begin{bmatrix}\cos\theta/2 \\ \sin\theta/2\, e^{i\phi}\end{bmatrix} \checkmark$

$|\psi_2> = \cos\frac{(\theta-\pi)}{2}|0> + \sin\frac{(\theta-\pi)}{2}e^{-i\phi}|1> = \begin{bmatrix}\cos\frac{(\theta-\pi)}{2} \\ \sin\frac{(\theta-\pi)}{2}e^{-i\phi}\end{bmatrix}$

(b) $(|\psi_1>, |\psi_2>) = <\psi_1|\psi_2> =$

$$\begin{bmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi}\end{bmatrix}\begin{bmatrix}\cos\frac{\theta-\pi}{2} \\ \sin\frac{\theta-\pi}{2}e^{-i\phi}\end{bmatrix}$$

$$= \cos\frac{\theta}{2}\cos\frac{(\theta-\pi)}{2} + \sin\frac{\theta}{2}\cdot\sin\frac{(\theta-\pi)}{2}\, e^{-i\phi}\cdot e^{-i\phi}$$

$$= \cos\frac{\theta}{2}\cos\frac{\theta-\pi}{2} + \sin\frac{\theta}{2}\sin\frac{(\theta-\pi)}{2}\, e^{-i\phi}\cdot e^{-i\phi}$$

$e^{i\phi}\cdot e^{-i\phi}$

$(\cos\phi + i\sin\phi)$
$(\cos\phi - i\sin\phi)$

$\cos^2\phi - i\cos +$
$i\sin\cdot + \sin^2\phi$
$= 1$

From this last equation it is clear that.

the $(e^{-i\phi}\, e^{-i\phi})$ term will not allow the $(|\psi_1\rangle, |\psi_2\rangle)$

to go to zero.

Hence, they are <u>NOT</u> orthogonal ✓

$(\cos\phi - i\sin\phi)$
$(\cos\phi - i\sin\phi)$
$\cos^2\phi - i\sin\phi\cos\phi$
$\mp\sin^2\phi$
$- i\sin\phi\cos\phi$
$\cos^2\phi - \sin^2\phi$
$- 2i\sin\phi\cos\phi$

(iii)

$$|\psi_1'\rangle = \cos\left(\frac{-\theta}{2}\right)|0\rangle + \sin\left(\frac{-\theta}{2}\right)e^{-i\phi}|1\rangle$$

$$|\psi_2'\rangle = \cos\frac{-(\theta-\pi)}{2}|0\rangle + \sin\frac{-(\theta-\pi)}{2}e^{i\phi}|1\rangle$$

$$|\psi_1'\rangle = \begin{bmatrix}\cos -\theta/2 \\ \sin(-\theta/2\, e^{-i\phi}\end{bmatrix} \qquad \alpha_1' = \cos(-\theta/2) \qquad \beta_1' = \sin(-\theta/2\, e^{-i\phi}$$

$$|\psi_2'\rangle = \begin{bmatrix}\cos\frac{-(\theta-\pi)}{2} \\ \sin\frac{-(\theta-\pi)}{2}e^{i\phi}\end{bmatrix} \qquad \alpha_2' = \cos\frac{-(\theta-\pi)}{2} \qquad \beta_2' = \sin\frac{-(\theta-\pi)}{2}e^{i\phi}.$$

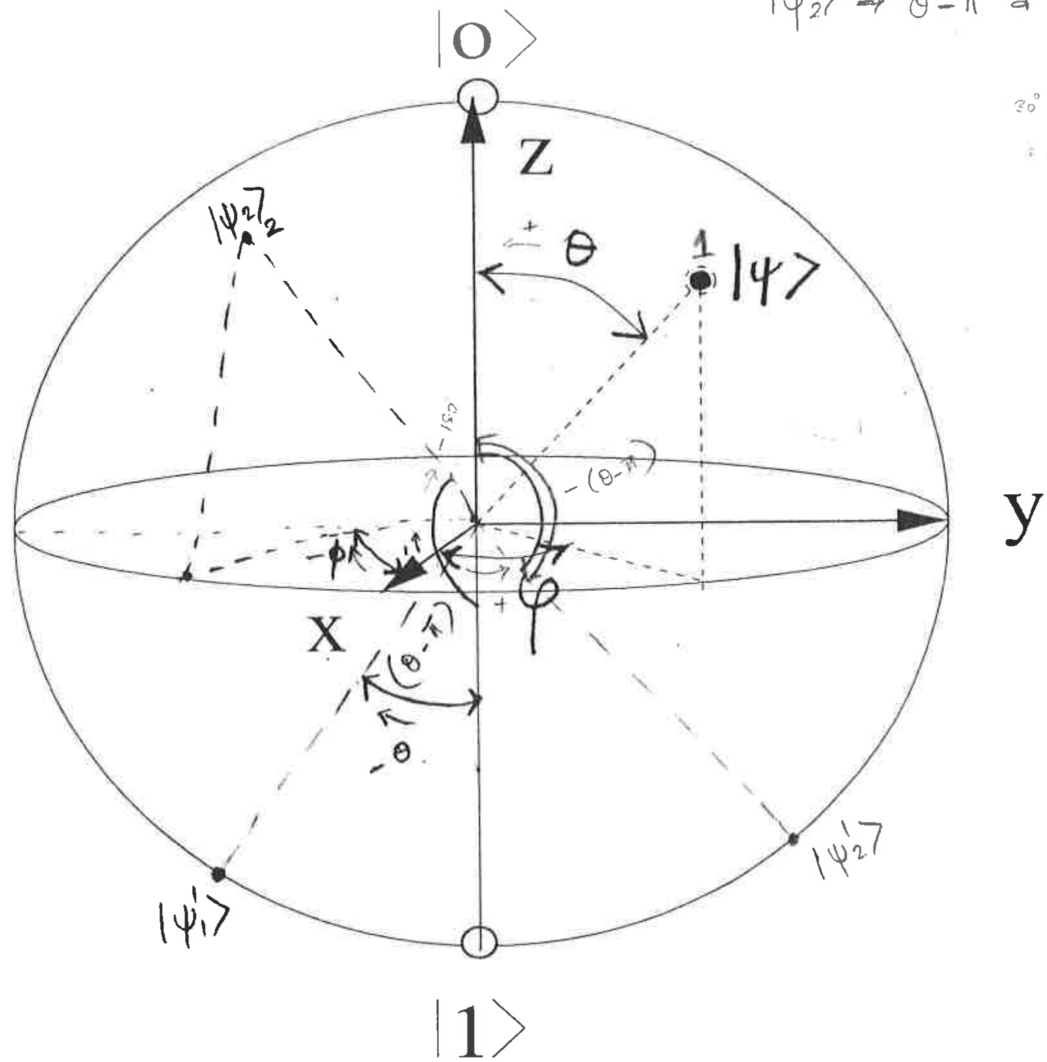
$-(\theta-\pi)$
$\cos(\pi-\theta)$
$= \sin\frac{\theta}{2}$.

$$|\psi_1'\rangle = \cos\frac{(-\theta)}{2}|0\rangle + \sin\frac{(-\theta)}{2}e^{-i\phi}|1\rangle = \sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{-i\phi}|1\rangle$$

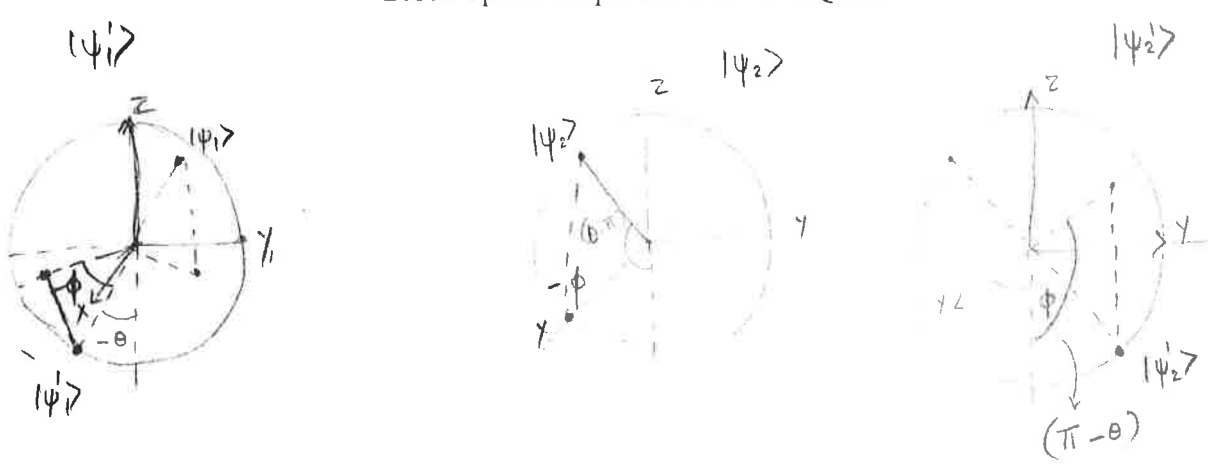
$$|\psi_2'\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{\mp i\phi}|1\rangle$$

$|\psi_1\rangle = \theta, \phi$

$|\psi_2\rangle \rightarrow \theta - \pi$ & $-\phi$



Bloch Sphere Representation of a Qubit



$(A \otimes B)(x \otimes z)$

$Ax \otimes Bz$

- (V) What is the average value of the tensor product

$$\sigma_x \otimes I \otimes \sigma_y$$

(where I is the 2x2 identity matrix) in the ket of the 3-qubit state

$$|0> \otimes |1> \otimes [\frac{1}{2}(|0> + |1>)]$$

$(A \otimes B)(x \otimes z)$

$Ax \otimes Bz$

$\langle 00| x \otimes I |11\rangle$

$(\langle 0| \otimes \langle 0|)(x \otimes I)$

$(|1\rangle \otimes |1\rangle)$

$\langle 0|x|1\rangle \ \langle 0|1\rangle$

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Average Value:-

$\langle \psi | (\sigma_x \otimes I \otimes \sigma_y) | \psi \rangle$

$$\left(\langle 0| \otimes \langle 1| \otimes \left[\frac{1}{2} \langle 0| + \langle 1| \right] \right) \left(\sigma_x \otimes I \otimes \sigma_y \right. \left. \left(|0\rangle \otimes |1\rangle \otimes \left[\frac{1}{2}(|0\rangle + |1\rangle) \right] \right) \right)$$

$\sigma_x \sigma_y = \sigma_y \sigma_x$

$\sigma_x \sigma_y - \sigma_y \sigma_x = 0$

$$= \left(\langle 0| \sigma_x |0\rangle \ \langle 1| I |1\rangle \ \left(\frac{1}{2} \langle 0| + \langle 1| \right) \sigma_y \left(\frac{1}{2} |0\rangle + |1\rangle \right) \right)$$

$$= \left((1\ 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \left((0\ 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \left(\frac{1}{4} \left[(1\ 1) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \right)$$

$$= \left[(1\ 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \left[(0\ 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \left[\frac{1}{4} \left[(1\ 1) \begin{pmatrix} -i \\ i \end{pmatrix} \right] \right]$$

$$= (0) \cdot (1) \left(\frac{1}{4} \left[\cancel{-i + i} \right] \right) = 0$$

Another way would be to break up the added Parts and do an entire Kronecker Product But, it would eventually be $= 0$ due to $\langle 0 | \sigma_x | 0 \rangle$.

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Kyle Brooks (M02653793)

Intro to Quantum Computing : ECES 622 - Winter 2011

Midterm Exam: Due Wednesday February 9, 2011 by 5p.m in my departmental mailbox.

(I) When we talked about the generalized Heisenberg Principle in class, we assume that if $|\psi>$ is a quantum state and A and B are two Hermitian operators, then with

$$<\psi|AB|\psi>= x+iy, \tag{1}$$

we have

$$<\psi|BA|\psi>= x-iy. \tag{2}$$

Prove this last equality starting with Eq.(1) and the fact that A and B are Hermitians.

$$<\psi|AB|\psi>^{*}= x-iy =<\psi|(AB)^{\dagger}|\psi>=<\psi|B^{\dagger}A^{\dagger}|\psi>=<\psi|BA|\psi>$$

(II) If A and B are two 2x2 matrices, show that

$$Tr(AB)=Tr(BA), \tag{3}$$

where Tr stands for the trace of the matrix. This is valid for any nxn square matrices but you are asked to show it for 2x2 matrices only.

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix};\ B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$$

$$BA=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} b_{11}a_{11}+b_{12}a_{21} & b_{11}a_{12}+b_{12}a_{22} \\ b_{21}a_{11}+b_{22}a_{21} & b_{21}a_{12}+b_{22}a_{22} \end{pmatrix}$$

$$Tr(AB)=Tr\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}=a_{11}b_{11}+a_{12}b_{21}+a_{21}b_{12}+a_{22}b_{22}$$

$$Tr(BA)=Tr\begin{pmatrix} b_{11}a_{11}+b_{12}a_{21} & b_{11}a_{12}+b_{12}a_{22} \\ b_{21}a_{11}+b_{22}a_{21} & b_{21}b_{12}+b_{22}a_{22} \end{pmatrix}=b_{11}a_{11}+b_{12}a_{21}+b_{21}a_{12}+b_{22}a_{22}$$

$$=a_{11}b_{11}+a_{12}b_{21}+a_{21}b_{12}+a_{22}b_{22}=Tr(AB)$$

(III) Show that for any three operators A,B,C, the following identity holds

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0. \tag{4}$$

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=[A,BC-CB]+[B,CA-AC]+[C,AB-BA]$$

$$=A(BC-CB)-(BC-CB)A+B(CA-AC)-(CA-AC)B+C(AB-BA)-(AB-BA)C$$

$$=\cancel{ABC}-\cancel{ACB}-\cancel{BCA}+\cancel{CBA}+\cancel{BCA}-\cancel{BAC}-\cancel{CAB}+\cancel{ACB}+\cancel{CAB}-\cancel{CBA}-\cancel{ABC}+\cancel{BAC}=0$$

(IV) Write the outer product representation of the Hadamard matrix H given below using the orthonormal basis formed of the eigenvectors of the Pauli matrix σ_x

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad (5)$$

$$|-\rangle_{x\,x}\langle -| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$|-\rangle_{x\,x}\langle +| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$|+\rangle_{x\,x}\langle -| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$|+\rangle_{x\,x}\langle +| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = x_4\left(|+\rangle_{x\,x}\langle +|\right) + x_3\left(|+\rangle_{x\,x}\langle -|\right) + x_2\left(|-\rangle_{x\,x}\langle +|\right) + x_1\left(|-\rangle_{x\,x}\langle -|\right)$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \xrightarrow{yields}$$

$$rref\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} \xrightarrow{yields} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \xrightarrow{yields}$$

$$\mathrm{H} = \frac{1}{\sqrt{2}}(|+>_x\,{}_x<+| + |+>_x\,{}_x<-| + |->_x\,{}_x<+| - |->_x\,{}_x<-|)$$

(V) Calculate the average value and standard deviation of the operator σ_x for a spin in the qubit state

$$|\psi>= \sqrt{\tfrac{1}{10}}(3|0> -|1>) \tag{6}$$

$$|\psi>= \sqrt{\frac{1}{10}}(3|0> -|1>) = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$$

$$S_X = \tfrac{\hbar}{2}\sigma_X$$

$$E(S_X) =< \psi|S_X|\psi >= \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}\frac{\hbar}{2} = -\frac{6\hbar}{10\cdot 2} = -\frac{3\hbar}{10}$$

$$\Delta(S_X) = \sqrt{E(S_X{}^2) - E(S_X)^2} = \sqrt{< \psi|S_X{}^2|\psi > - \left(-\frac{3\hbar}{10}\right)^2}$$

$$< \psi|S_X{}^2|\psi >= \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} I \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}\frac{\hbar^2}{4}$$

$$= \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}\frac{\hbar^2}{4} = \left(\frac{9}{10} + \frac{1}{10}\right)\frac{\hbar^2}{4} = \frac{\hbar^2}{4}$$

$$\Delta(S_x) = \sqrt{\frac{\hbar^2}{4} - \frac{9\hbar^2}{100}} = \sqrt{\frac{25\hbar^2}{100} - \frac{9\hbar^2}{100}} = \sqrt{\frac{16\hbar^2}{100}} = \frac{4\hbar}{10} = \frac{2\hbar}{5}$$

(VI)What is the probability to find $\frac{\hbar}{2}$ when measuring the x-component of the spin , i.e., σ_x when a qubit is prepared in the quantum state given by Eq.(6)?

$$|\psi> = \sqrt{\frac{1}{10}}(3|0> - |1>) = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$$

$$P\left(\frac{\hbar}{2}\right) = |<\psi|+>_x|^2 = \left|\begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right|^2 = \left|\frac{3}{\sqrt{20}} - \frac{1}{\sqrt{20}}\right|^2 = \left|\frac{2}{\sqrt{20}}\right|^2 = \frac{4}{20} = \frac{2}{10} = 0.2$$

(VII) While covering the principles of Quantum Mechanics, we showed that the connection between solutions to the Schroedinger equation between time t_1 and t_2 for a system with Hamiltonian H is given by the operator

$$U = exp\left(-\frac{i}{\hbar}H(t_2 - t_1)\right) \tag{7}$$

where $\hbar = \frac{h}{2}$ with h being Planck's constant.

Show that the operator U is unitary. Hint: the Hamiltonian H is Hermitian and therefore normal. It has a spectral decomposition, i.e.

$$H = \sum E_i|E_i >< E_i| \tag{8}$$

where E_i are the eigenvalues of H with corresponding eigenstate $|E_i>$. Use that to write the spectral decomposition of U, and then prove it is unitary.

$$U = exp\left(-\frac{i}{\hbar}H(t_2 - t_1)\right) = U = exp\left(-\frac{i}{\hbar}\sum E_i|E_i >< E_i|(t_2 - t_1)\right)$$

U is normal so we can apply the spectral decomposition theorem:

$$U = exp\left(-\frac{i}{\hbar}\sum E_i|E_i >< E_i|(t_2 - t_1)\right) = \sum exp\left(-\frac{i}{\hbar}E_i(t_2 - t_1)\right)|E_i >< E_i|$$

$$U^* = \sum exp\left(\frac{i}{\hbar}E_j^*(t_2 - t_1)\right)|E_j^* >< E_j^*|$$

When U is multiplied by its Hermitian conjugate the exponents and orthogonal components cancel ie:

$$exp\left(-\frac{i}{\hbar}E_i(t_2-t_1)\right)exp\left(\frac{i}{\hbar}E_i^*(t_2-t_1)\right)|E_i><E_i|\,|E_j^*><E_j^*| = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

This is because:

$$UU^* = \sum|E_i><E_i|\sum|E_j^*><E_j^*|$$

$$U^*U = \sum|E_j^*><E_j^*|\sum|E_i><E_i|$$

and by definition the vectors of $<E_i|$ and $<E_j^*|$ form an orthonormal basis spanning the system with the Hamiltonian H thus:

$$\sum|E_i><E_i|\sum|E_j^*><E_j^*| = \sum|E_j^*><E_j^*|\sum|E_i><E_i| = I^2 = I$$

For the system with Hamiltonian H and:

$$UU^* = U^*U = I \xrightarrow{yields}$$

U is a unitary operator.

(VIII) Show that 2x2 any matrix M can be written as follows

$$M = a_0 I + \vec{a}\cdot\vec{\sigma} \tag{9}$$

where I is the 2x2 identity matrix and

$$a_0 = \tfrac{1}{2}Tr(M) \tag{10}$$

$$\vec{a} = \tfrac{1}{2}Tr(\vec{\sigma}M) \qquad \textbf{Corrected } (11)$$

$$M = \begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix} \in \mathbb{C}^2 \xrightarrow{yields} a_0 = \frac{1}{2}Tr(M) = \frac{1}{2}(m_{11}+m_{22})$$

$$\vec{a} = a_x + a_y + a_z$$

$$a_x = \frac{1}{2}Tr(\sigma_x M) = \frac{1}{2}Tr\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\right) = \frac{1}{2}Tr\left(\begin{pmatrix} m_{21} & m_{22}\\ m_{11} & m_{12}\end{pmatrix}\right) = \frac{1}{2}(m_{21}+m_{12})$$

$$a_y = \frac{1}{2}Tr(\sigma_y M) = \frac{1}{2}Tr\left(\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\right) = \frac{1}{2}Tr\left(\begin{pmatrix} -m_{21}i & -m_{22}i\\ m_{11}i & m_{12}i\end{pmatrix}\right) = \frac{1}{2}(-m_{21}i + m_{12}i)$$

$$a_z = \frac{1}{2}Tr(\sigma_z M) = \frac{1}{2}Tr\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\right) = \frac{1}{2}Tr\left(\begin{pmatrix} m_{11} & m_{12}\\ -m_{21} & -m_{22}\end{pmatrix}\right) = \frac{1}{2}(m_{11}-m_{22})$$

$$M = a_0 I + \vec{a}\cdot\vec{\sigma} = a_0 I + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$$

$$a_0 I = \frac{1}{2}(m_{11}+m_{22})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}(m_{11}+m_{22}) & 0\\ 0 & \frac{1}{2}(m_{11}+m_{22})\end{pmatrix}$$

$$a_x\sigma_x = \frac{1}{2}(m_{21}+m_{12})\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{2}(m_{21}+m_{12})\\ \frac{1}{2}(m_{21}+m_{12}) & 0\end{pmatrix}$$

$$a_y\sigma_y = \frac{1}{2}(-m_{21}i+m_{12}i)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{2}(-m_{21}+m_{12})\\ \frac{1}{2}(m_{21}-m_{12}) & 0\end{pmatrix}$$

$$a_z\sigma_z = \frac{1}{2}(m_{11}-m_{22})\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}(m_{11}-m_{22}) & 0\\ 0 & \frac{1}{2}(-m_{11}+m_{22})\end{pmatrix}$$

$$M = \begin{pmatrix}\frac{1}{2}(m_{11}+\cancel{m_{22}}) & 0\\ 0 & \frac{1}{2}(\cancel{m_{11}}+m_{22})\end{pmatrix} + \begin{pmatrix}0 & \frac{1}{2}(\cancel{m_{21}}+m_{12})\\ \frac{1}{2}(m_{21}+\cancel{m_{12}}) & 0\end{pmatrix}$$
$$+\begin{pmatrix}0 & \frac{1}{2}(-\cancel{m_{21}}+m_{12})\\ \frac{1}{2}(m_{21}-\cancel{m_{12}}) & 0\end{pmatrix} + \begin{pmatrix}\frac{1}{2}(m_{11}-\cancel{m_{22}}) & 0\\ 0 & \frac{1}{2}(-\cancel{m_{11}}+m_{22})\end{pmatrix} \xrightarrow{yields}$$

$$M = \begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}$$

(IX) Prove starting with the general spinor to sweep the Bloch sphere,

$$|\xi_n{}^+ >= \cos\left(\frac{\theta}{2}\right)|0> + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1>$$

the following is true

$$<\xi_n{}^+|\sigma_x|\xi_n{}^+>= \sin\theta\cos\phi \tag{12}$$

$$<\xi_n{}^+|\sigma_y|\xi_n{}^+>= \sin\theta\sin\phi \tag{13}$$

and

$$<\xi_n{}^+|\sigma_z|\xi_n{}^+>= \cos\theta \tag{14}$$

$$|\xi_n{}^+ >= \cos\left(\frac{\theta}{2}\right)\begin{pmatrix}1\\0\end{pmatrix} + \sin\left(\frac{\theta}{2}\right)e^{i\phi}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{pmatrix}$$

$$< \xi_n{}^+|\sigma_x|\xi_n{}^+ >= \left(\cos\left(\frac{\theta}{2}\right) \quad \sin\left(\frac{\theta}{2}\right)e^{i\phi}\right)\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{pmatrix}$$

$$= \left(\sin\left(\frac{\theta}{2}\right)e^{i\phi} \quad \cos\left(\frac{\theta}{2}\right)\right)\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{pmatrix}$$

$$= \sin\left(\frac{\theta}{2}\right)e^{i\phi}\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{-i\phi} = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(e^{i\phi} + e^{-i\phi}\right)$$

$$2\cos(x) = e^{ix} + e^{-ix};\ \sin\left(\frac{x}{2}\right) = \sqrt{\frac{1-\cos(x)}{2}};\ \cos\left(\frac{x}{2}\right) = \sqrt{\frac{1+\cos(x)}{2}} \xrightarrow{yields}$$

$$< \xi_n{}^+|\sigma_x|\xi_n{}^+ >= \sqrt{\frac{1-\cos(\theta)}{2}}\sqrt{\frac{1+\cos(\theta)}{2}}\,2\cos(\phi) = \sin\theta\cos\phi$$

$$< \xi_n{}^+|\sigma_y|\xi_n{}^+ >= \left(\cos\left(\frac{\theta}{2}\right) \quad \sin\left(\frac{\theta}{2}\right)e^{i\phi}\right)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{pmatrix}$$

$$= \left(\sin\left(\frac{\theta}{2}\right)e^{i\phi}i \quad -\cos\left(\frac{\theta}{2}\right)i\right)\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{pmatrix}$$

$$= \sin\left(\frac{\theta}{2}\right)e^{i\phi}\cos\left(\frac{\theta}{2}\right)i - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{-i\phi}i = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)i\left(e^{i\phi} - e^{-i\phi}\right)$$

$$2i\sin(x) = e^{ix} - e^{-ix};\ \sin\left(\frac{x}{2}\right) = \sqrt{\frac{1-\cos(x)}{2}};\ \cos\left(\frac{x}{2}\right) = \sqrt{\frac{1+\cos(x)}{2}} \xrightarrow{yields}$$

$$< \xi_n{}^+|\sigma_x|\xi_n{}^+ >= \sqrt{\frac{1-\cos(\theta)}{2}}\sqrt{\frac{1+\cos(\theta)}{2}}\,i2i\sin(\phi) = \sin\theta\sin\phi$$

$$< \xi_n{}^+ | \sigma_z | \xi_n{}^+ > = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix} = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \cos(\theta)$$

(X) Show that the qubit flip matrix M

$$M = P(\phi) X P(\pi - \phi) \tag{15}$$

where $P(\phi)$ is the phase shift matrix and X is the Pauli matrix, is unitary.

$$P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

$$M = P(\phi) X P(\pi - \phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\pi-\phi)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ e^{i\phi} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\pi-\phi)} \end{pmatrix} = \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

$$MM^\dagger = \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix} = \begin{pmatrix} (-e^{-i\phi})(-e^{i\phi}) & 0 \\ 0 & e^{i\phi} e^{-i\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$M^\dagger M = \begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix} \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} = \begin{pmatrix} e^{-i\phi} e^{i\phi} & 0 \\ 0 & (-e^{i\phi})(-e^{-i\phi}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$M^\dagger M = MM^\dagger = I \xrightarrow{yields} M \text{ is unitary}$$